ZESZYTY NAUKOWE POLITECHNIKI RZESZOWSKIEJ 290, Mechanika 86
RUTMech, t. XXXI, z. 86 (1/14), styczeń-marzec 2014, s. 5-15

Sergei ALEXANDROV[1]
Elena LYAMINA[2]
Nguyen Minh TUAN[3]

AN UPPER BOUND SOLUTION FOR COMPRESSION OF VISCOUS MATERIAL BETWEEN ROTATING PLATES

An upper bound solution for compression of viscous material between rotating plates is proposed. For many conventional constitutive equations its form has been given by Hill. In the case of viscous materials the main difficulty with the application of the upper bound theorem is that conventional friction laws are not compatible with the conditions used to prove it. A reduced version of the upper bound theorem that accounts for specific viscous constitutive equations and boundary conditions is adopted. In such a form, in contrast to the general case, the theorem determines an upper bound on the load required to deform the material. The dependence of the upper bound force based on a simple kinematically admissible velocity field on material and process parameters is illustrated. The solution is reduced to numerical integration and minimization of a function of one variable.

Keywords: upper bound, friction, metal forming, viscoplasticity.

1. Introduction

The upper bound theorem is a convenient tool for finding approximate rigid plastic solutions, in particular in material forming applications. Most of such solutions are based on rigid perfectly plastic material models [1, 2]. In this case the functional for minimization is the plastic work rate and a typical result of calculations is an upper bound on the load required to deform the material. In general, the functional involved in the upper bound theorem depends on the constitutive equations chosen. For many conventional constitutive equations its

[1] Autor do korespondencji/corresponding author: Sergei ALEXANDROV, Ishlinskii Institute for Problems in Mechanics, Russian Academy of Sciences, 101-1 Prospect Vernadskogo, 119526 Moscow, Russia; e-mail: sergei_alexandrov@spartak.ru
[2] Elena LYAMINA, Russian Academy of Sciences, 101-1 Prospect Vernadskogo, 119526 Moscow, Russia
[3] Nguyen Minh TUAN, Vietnam Academy of Sciences and Technology, 264 Doi Can, Ba Dinh, Ha Noi, Vietnam

form has been given in [3]. In the case of viscous materials the main difficulty with the application of the upper bound theorem is that conventional friction laws are not compatible with the conditions used to prove it. This kind of difficulties is reviewed and explained in [4]. Nevertheless, the requirements of the upper bound theorem for viscous materials are often ignored [5 - 7 among many others]. The correct formulation has been adopted in [8, 9]. Moreover, even if the upper bound theorem for viscous materials is applicable, it does not lead, in general, to an upper bound on the load applied. However, it has been shown in [10] that there is a class of processes where the theorem does provide an upper bound on the load applied. The additional requirements in this case are: (i) maximum friction law, (ii) stress free boundary conditions on the entire surface of a deforming body except the friction surface, and (iii) viscous power-law material. The maximum friction law postulates that the friction stress is equal to the shear yield stress at a given magnitude of the equivalent strain rate. Its use leads to the regime of sticking [11] and therefore the upper bound theorem becomes applicable [10]. The upper bound theorem with the superimposed restrictions (i) to (iii) has been applied to analyze several axisymmetric processes [10, 12]. In the present paper the theorem is adopted to find an upper bound load for plane strain compression of a block between rotating plates. Such a process is of interest for practical applications [13].

2. Statement of the problem

A schematic diagram of the process is shown in Fig.1. A block of viscous material is compressed between two plates rotating with an angular velocity ω. It is convenient to introduce two coordinate systems, namely a polar coordinate system (r,θ) and a Cartesian coordinate system (x, y), as shown in Fig.1. The plates rotate around the origin of the coordinate systems. The surfaces $x = R_0 \cos\theta_0$ and $x = R_0 \cos\theta_0 + L$ are traction free. The maximum friction law is assumed at $\theta = \pm\theta_0$. Because of symmetry, it is sufficient to consider the domain $\theta \geq 0$ (or $y \geq 0$). Then, the velocity boundary conditions are

$$u_\theta = -\omega r \tag{1}$$

at $\theta = \theta_0$,

$$u_\theta = 0 \tag{2}$$

at $\theta = 0$ and

$$u_r = 0 \tag{3}$$

at $\theta = \theta_0$. As has been mentioned before, the condition (3) is equivalent to the maximum friction law in the case of viscous power-law materials. In (1) to (3), u_r and u_θ are the radial and circumferential velocities, respectively.

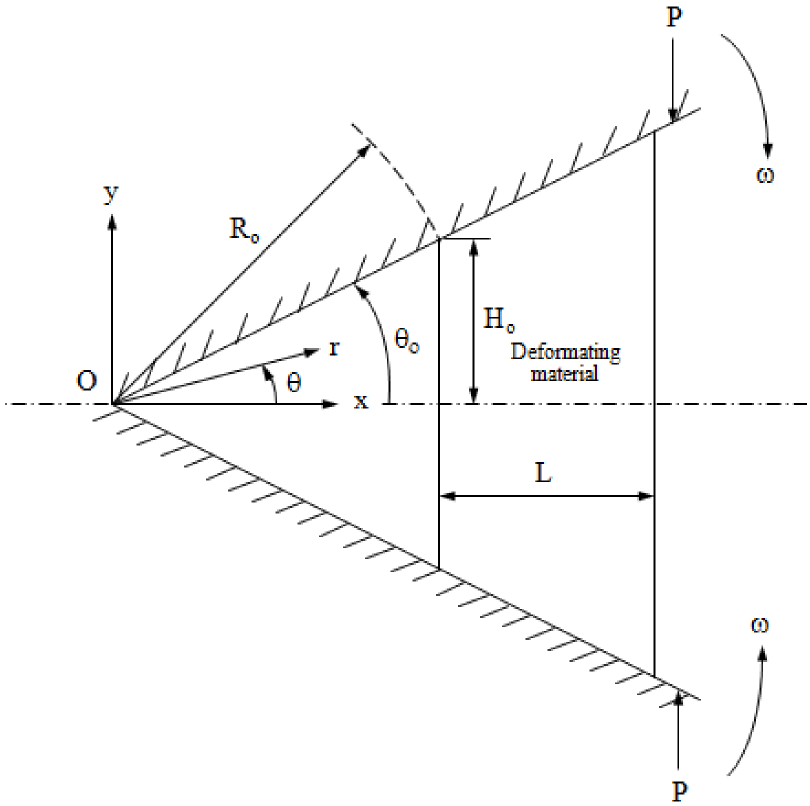

Fig. 1. Schematic diagram of the process

The material model is defined by the following equations: $\xi_{ij} = \lambda \tau_{ij}$ and $\sigma_{eq} = K \xi_{eq}^n$. Here, ξ_{ij} are the components of the strain rate tensor, τ_{ij} are the deviatoric components of the stress tensor, ξ_{eq} is the equivalent strain rate, σ_{eq} is the equivalent stress, λ is a non-negative multiplier, K is a rheological constant, and n is the strain rate sensitivity exponent. The equivalent strain rate and equivalent stress are defined by

$$\xi_{eq} = \sqrt{(2/3)\,\xi_{ij}\xi_{ij}}\,, \qquad \sigma_{eq} = \sqrt{(3/2)\,\tau_{ij}\tau_{ij}}\,. \tag{4}$$

For the problem under consideration the upper bound theorem reads [4]

$$M\omega \le K\iiint_V \xi_{eq}^{n+1}\,dV \tag{5}$$

where *M* is the moment of force *P* (Fig.1) and ξ_{eq} should be calculated with the use of a kinematically admissible velocity field. The latter must satisfy the incompressibility equation and the boundary conditions (1) to (3). Additional conditions following from symmetry are that u_θ is an odd function of θ and u_r is an even function of θ. Even though those are not necessary conditions when the upper bound theorem is adopted to solve the problem, it is advantageous to account for these conditions in kinematically admissible velocity fields. Assume that

$$\frac{u_\theta}{\omega r} = -\frac{\theta}{\theta_0}\left[1 + f(r)\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right)\right] \tag{6}$$

where $f(r)$ is an arbitrary function of *r*. The circumferential velocity in the form of (6) satisfies the boundary conditions (1) and (2) as well as the additional condition that u_θ is an odd function of θ. The terms in the brackets can be understood as the first two terms of a Fourier expansion of an arbitrary function of r and θ satisfying the aforementioned conditions. The incompressibility equation in polar coordinates is

$$\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_\theta}{\partial\theta} = 0. \tag{7}$$

Substituting (6) into (7) and integrating give

$$u_r = \frac{\omega\left(r^2 - R^2\right)}{2\theta_0 r} + \frac{\omega}{\theta_0 r}\left[\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right) - \frac{\pi}{2}\frac{\theta}{\theta_0}\sin\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right)\right]\int_R^r rf(r)\,dr. \tag{8}$$

Here *R* is in general a function of θ. However, in the present solution it is supposed that $R = \text{constant}$. Combining (8) and the boundary condition (3) leads to

$$\left(r^2 - R^2\right) - \pi \int_R^r r f\left(r\right) dr = 0 . \tag{9}$$

It follows from this equation that $f\left(r\right) = 2/\pi$. Then, equations (6) and (8) transform to

$$\frac{u_\theta}{\omega r} = -\frac{\theta}{\theta_0}\left[1 + \frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right)\right]. \tag{10}$$

$$\frac{u_r}{\omega r} = \frac{1}{2\theta_0}\left[1 + \frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right) - \frac{\theta}{\theta_0}\sin\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right)\right]\frac{\left(r^2 - R^2\right)}{r^2} . \tag{11}$$

Equations (10) and (11) provide a kinematically admissible velocity field.

3. Upper Bound Theorem

In the case of plane strain deformation the definition for the equivalent strain rate (4) written in the polar coordinates simplifies to

$$\xi_{eq} = \frac{2}{\sqrt{3}}\sqrt{\xi_{rr}^2 + \xi_{r\theta}^2} . \tag{12}$$

It has been taken into account here that $\xi_{rr} = -\xi_{\theta\theta}$ due to the incompressibility equation. The radial and shear strain rates can be calculated by means of (10) and (11) with no difficulty. In particular,

$$\xi_{r\theta} = -\frac{\omega}{4\theta_0^2}\left(1 - \frac{R^2}{r^2}\right)\left[2\sin\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right) + \frac{\pi}{2}\frac{\theta}{\theta_0}\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right)\right]. \tag{13}$$

$$\xi_{rr} = \frac{\omega}{2\theta_0}\left(1 + \frac{R^2}{r^2}\right)\left[1 - \frac{\theta}{\theta_0}\sin\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right) + \frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right)\right]. \tag{14}$$

Substituting (13) and (14) into (12) gives

$$\xi_{eq} = \frac{\omega}{\sqrt{3}\theta_0^2}\sqrt{\theta_0^2\left(1+\frac{R^2}{r^2}\right)^2 \zeta_{rr}^2(\theta) + \frac{1}{4}\left(1-\frac{R^2}{r^2}\right)^2 \zeta_{r\theta}^2(\theta)}$$

$$\zeta_{rr}(\theta) = 1 - \frac{\theta}{\theta_0}\sin\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right) + \frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right), \tag{15}$$

$$\zeta_{r\theta}(\theta) = 2\sin\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right) + \frac{\pi}{2}\frac{\theta}{\theta_0}\cos\left(\frac{\pi}{2}\frac{\theta}{\theta_0}\right)$$

Using (15) the inequality (5) can represented in the form

$$M^u\omega = BKR_0^2\left(\frac{\omega}{\sqrt{3}\theta_0^2}\right)^{n+1}\int_0^{\theta_0}\int_{\rho_1(\theta)}^{\rho_2(\theta)}\left[\begin{array}{l}\theta_0^2\left(1+\frac{\beta}{\rho^2}\right)^2\zeta_{rr}^2(\theta)+\\ +\frac{1}{4}\left(1-\frac{\beta}{\rho^2}\right)^2\zeta_{r\theta}^2(\theta)\end{array}\right]^{\frac{n+1}{2}}\rho d\rho d\theta . \tag{16}$$

Here M^u is an upper bound on M, B is the thickness of the block, $\rho = r/R_0$, $\beta = R^2/R_0^2$, $\rho_1(\theta)$ and $\rho_2(\theta)$ are determined from the equations $x = R_0\cos\theta_0$ and $x = R_0\cos\theta_0 + L$ in the form

$$\rho_1(\theta) = \frac{\cos\theta_0}{\cos\theta}, \qquad \rho_2(\theta) = \frac{\cos\theta_0 + l}{\cos\theta} \tag{17}$$

where $l = L/R_0$. It is convenient to introduce the dimensionless upper bound moment by

$$m^u = \frac{3^{\frac{n+1}{2}}\theta_0^{2(n+1)}M^u}{BK\omega^n R_0^2}. \tag{18}$$

It follows from (16) and (18) that

$$m^u = \int_0^{\theta_0}\int_{\rho_1(\theta)}^{\rho_2(\theta)} \left[\begin{array}{l} \theta_0^2\left(1+\dfrac{\beta^2}{\rho^2}\right)^2 \zeta_{rr}^2(\theta) + \\ +\dfrac{1}{4}\left(1-\dfrac{\beta^2}{\rho^2}\right)^2 \zeta_{r\theta}^2(\theta) \end{array}\right]^{\frac{n+1}{2}} \rho d\rho d\theta . \tag{19}$$

The right hand side of this equation should be minimized with respect to β to obtain the best upper bound based on the kinematically admissible velocity field chosen. It is obvious from the definition for β ,that $\beta > 0$.

4. Numerical Results

Minimization in (19) has been completed numerically. The variation of the dimensionless moment m^u with n, θ_0 and l is illustrated in Figs.2 to 4. The variation of β is also of some interest because its magnitude determines the sense of the shear strain rate $\xi_{r\theta}$, in particular at points of the friction surface. It follows from the constitutive equations that the shear stress has the same sense. Therefore, it finally determines the direction of the friction stress. In particular, equation (13) shows that $\xi_{r\theta} = 0$ at $\rho = \rho_n = \sqrt{\beta}$, $\xi_{r\theta} < 0$ in the interval $\rho > \rho_n$, and $\xi_{r\theta} > 0$ in the interval $\rho < \rho_n$. Therefore, if $1 < \rho_n < 1 + l/\cos\theta_0$, the friction stress is directed to the axis of rotation in the interval $\rho > \rho_n$ and from this axis in the interval $\rho < \rho_n$. If $1 > \rho_n$, the friction stress is directed to the axis of rotation over the entire friction surface, and if $1 + 1/\cos\theta_0 < \rho_n$ it is directed from the axis of rotation over the entire friction surface. However, the latter inequality is not satisfied in the cases considered. In particular, the variation of ρ_n with n, θ_0 and l is illustrated in Figs.5 to 7. The dash line corresponds to $\rho_n = 1$. Thus, above this line $\rho_n > 1$ and in such processes the neutral point actually exists at the friction surface. In the other case the friction stress is directed to the axis of rotation over the entire friction surface.

5. Conclusions

A new upper bound solution for viscous material compressed between two rotating plates has been found. A reduced form of the upper bound theorem has been adopted allowing for the determination of an upper bound on the force applied. The dimensionless representation of the moment of this force is independent of the angular velocity of plates, though the force of course is. The solution is illustrated in Figs. 2 to 7. The dependence of the moment on material and process parameters (Figs.2 to 4) is in agreement with physical expectations. In particular, the moment increases as l increases. The dimensionless moment increases as θ_0 increases. However, it is seen from (18) that M^u decreases. It also follows from (18) that M^u is an increasing function of both ω and K. The variation of ρ_n with material and process parameters (Figs.5 to 7) shows that the friction stresses may or may not change its direction. In the latter case, the friction stress is directed to the axis of rotation. It is interesting to mention that $\rho_n = 0$ at $\theta_0 = \pi/4$, i.e. in this case the strain rate components are independent of r, as follows from (13) and (14).

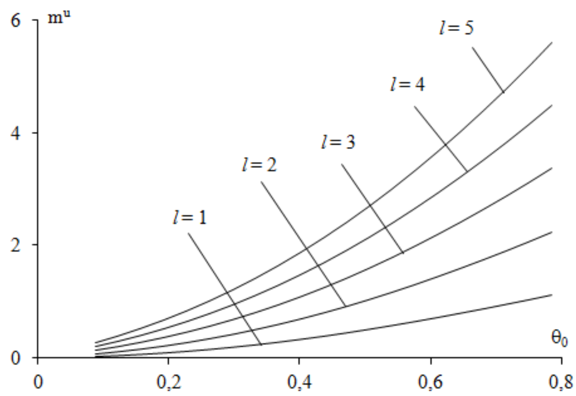

Fig. 2. Variation of the dimensionless moment with q_0 at different values of l and n = 0.1

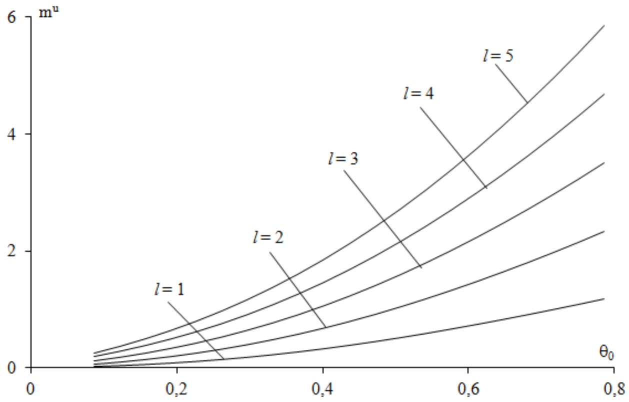

Fig. 3. Variation of the dimensionless moment with q_0 at different values of *l* and *n* = 0.3

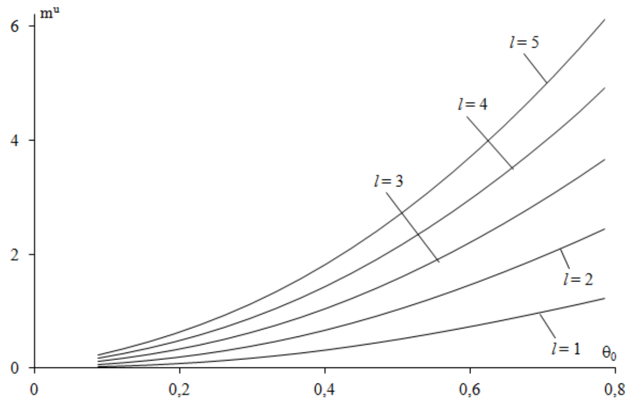

Fig. 4. Variation of the dimensionless moment with q_0 at different values of *l* and *n* = 0.5

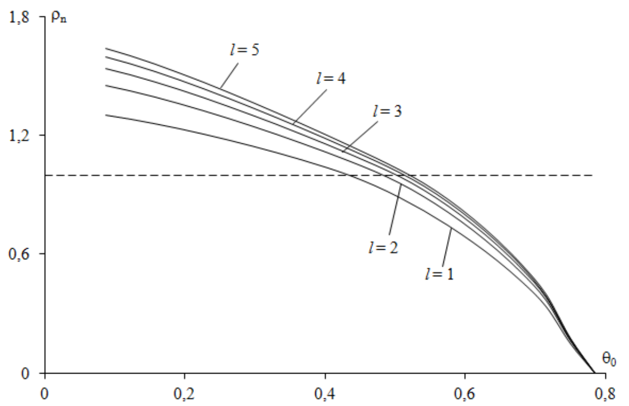

Fig. 5. Variation of the position of neutral point with q_0 at different values of *l* and *n* = 0.1

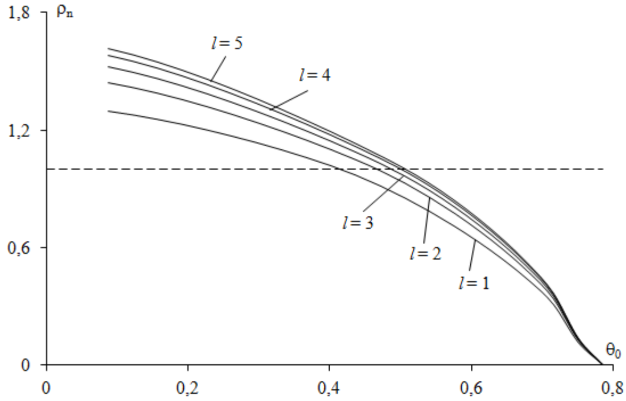

Fig. 6. Variation of the position of neutral point with q_0 at different values of *l* and n = 0.3

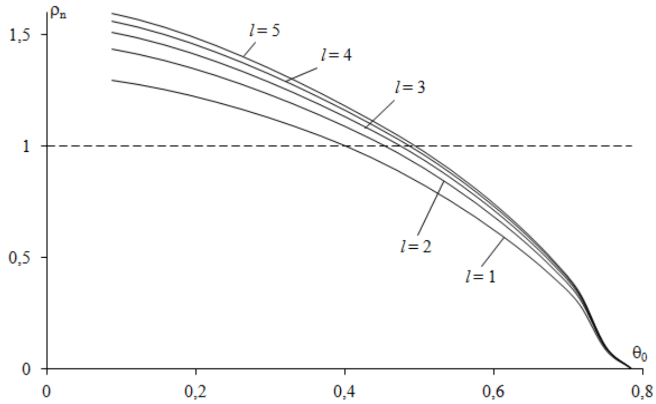

Fig. 7. Variation of the position of neutral point with q_0 at different values of *l* and n = 0.5

References

[1] Avitzur B.: Metal forming: the application of limit analysis, Dekker, New York 1980.

[2] Hill R. : New horizons in the mechanics of solids, J. Mech. Phys. Solids, 5 (1956) 66-74.

[3] Hill R.: A general method of analysis for metalworking processes, J. Mech. Phys. Solids, 11 (1963) 305-326.

[4] Alexandrov S.: An analysis of the axisymmetric compression of viscous materials. J. Mater. Proc. Technol., 105 (2000) 278-283.

[5] Alexandrov S., Danilov V., Chikanova N.: On the stagnation zone in a simulation of axisymmetric pressure metal forming under creep, Mech. Solids, 35 (2000) 127-129.

[6] Tzou G.-Y., Alexandrov S.: An upper bound solution for a solid viscous cylinder subject to compression and twist, J. Mater. Proc. Technol., 177 (2006) 159-162.

[7] Contractor D.N.: An inviscid analysis in polar coordinates of flow between two flat plates, ASME J. Appl. Mech., 60 (1993) 65-69.

Acknowledgment

The research described in this paper has been supported by grants RFBR-13-01-93000 (Russia) and VAST.HTQT.Nga.05/13-14 (Vietnam).

METODA GÓRNEJ OCENY W ZASTOSOWANIU DO ŚCISKANIA LEPKIEGO MATERIAŁU POMIĘDZY OBRACAJĄCYMI SIĘ PŁYTAMI

Streszczenie

W artykule zaproponowano wykorzystanie metody górnej oceny do opisu ściskania lepkiego materiału pomiędzy obracającymi sie płytami. Dla wielu konwencjonalnych równań konstytutywnych ich postać została przedstawiona przez Hill'a. W przypadku materiałów lepkich głównym problemem przy wykorzystaniu twierdzenia górnej granicy jest to, że klasyczne prawa tarcia nie są kompatybilne z warunkami stosowanymi do jego udowodnienia. Zastosowano uproszczoną wersję twierdzenia górnej granicy, uwzględniającą specyficzne równania konstytutywne materiałów lepkich oraz warunki brzegowe. W takiej postaci, w przeciwieństwie do przypadku ogólnego, twierdzenie określa górną granicę obciążenia wymaganą do odkształcenia materiału. Przedstawiono wpływ siły określonej za pomocą twierdzenia górnej granicy na podstawie kinematycznie dopuszczalnego pola prędkości na materiał i parametry procesu. Rozwiązanie zredukowano do całkowania numerycznego i minimalizacji funkcji jednej zmiennej.

Słowa kluczowe: górna granica, tarcie, obróbka plastyczna, wiskoplastyczność

DOI: 10.7862/rm.2014.1

Otrzymano/received: 15.01.2014
Zaakceptowano/accepted: 22.03.2014