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Fourier, Laguerre, Laplace Transforms with Applications

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ABSTRACT: In this article, the author considered certain time fractional equations using joint integral transforms. Transform method is a powerful tool for solving singular integral equations, integral equation with retarded argument, evaluation of certain integrals and solution of partial fractional differential equations. The obtained results reveal that the transform method is very convenient and effective. Illustrative examples are also provided.

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1. Introduction and Preliminaries

This article is devoted to the study of the Laguerre transforms and its basic operational properties. The joint Laplace-Laguerre transform can be used effectively to solve the heat conduction problem in a semi-infinite medium with variable thermal conductivity. Another application of the Laguerre transform is to solve the problem of oscillations of a very heavy chain with variable tension. Apart from the ordinary or partial derivatives which occur in elementary calculus various other types of derivatives are known in the fractional calculus literature. Examples of this type are, the Caputo fractional derivative, the Riemann-Liouville fractional derivative. In the last three decades, considerable progress has been made in the area of fractional derivatives and, in general, in the area of fractional calculus. Partial fractional differential equations play an important role in science, engineering and social sciences. Nowadays it is impossible to describe a viscoelastic process without using a fractional derivative. Many phenomena in fluid mechanics, physics, biology, engineering and other areas of the sciences can be successfully modeled by the use of fractional derivatives. At this point, it should be emphasized that several definitions have been proposed the fractional derivatives, among those the Caputo and Riemann-Liouville is the most popular. Among scientists and engineers the Caputo fractional derivatives are more popular. Fractional differential equations arise in the unification of wave and diffusion phenomenon. The time fractional heat conduction equation, which is a mathematical model of a wide range of important physical phenomena, is obtained from the classical heat equation by replacing the first time derivative by a fractional derivative of order $0 < \alpha < 1$.

1.1. Definitions and Notations

Definition 1.1. Fourier transform of the function $\psi(x)$ is defined as follows

$$\mathcal{F}\{\psi(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx}\psi(x)dx := \Psi(w). \tag{1.1}$$

If $\mathcal{F}\{\psi(x)\} = \Psi(w)$, then $\mathcal{F}^{-1}\{\Psi(w)\}$ is given by

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixw}\Psi(w)dw, \tag{1.2}$$

where $\psi(x)$, $\Psi(w)$ are elements of the $\mathcal{S}(R)$, space of rapidly decreasing functions or Schwartz class[13].

Note. The vector space $\mathcal{S}(R)$ of the rapidly decreasing functions is closed under linear combinations and differentiation. Any function belongs to $\mathcal{S}(R)$ is integrable. A typical function in this space is $\exp(-|x|^2)$.

Lemma 1.1. (Convolution Theorem for the Fourier transform)
Let us assume that $\mathcal{F}[\phi(x)] = \Phi(w)$ and $\mathcal{F}[\psi(x)] = \Psi(w)$, then the convolution of two functions $\phi(x)$ and $\psi(x)$ is defined by the expression

$$\phi(x) * \psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(\xi)\psi(x-\xi)d\xi,$$

and the Fourier transform of the convolution is as follows

$$\mathcal{F}[\phi(x) * \psi(x); x \to w] = \Phi(w)\Psi(w).$$

Lemma 1.2. *We have the following identities for the Fourier transform*

$$1. \qquad \mathcal{F}[e^{-a^2x^2}; x \to w] = \frac{1}{a\sqrt{2}} e^{-\frac{w^2}{4a^2}}.$$

$$2. \qquad \mathcal{F}[|x|^{-\alpha}; x \to w] = \sqrt{\frac{2}{\pi}} \frac{\Gamma(1-\alpha)}{|w|^{1-\alpha}} \sin(\frac{\pi\alpha}{2}).$$

$$3. \qquad \mathcal{F}[sgn(x); x \to w] = \sqrt{\frac{2}{\pi}}\frac{i}{w}.$$

$$4. \qquad \mathcal{F}[xsgn(x); x \to w] = -\sqrt{\frac{2}{\pi}}\frac{1}{w^2}.$$

$$5. \qquad \mathcal{F}[e^{-a|x|}; x \to w] = \sqrt{\frac{2}{\pi}}\frac{a}{w^2+a^2}.$$

Proof. See [4, 9, 13].

Lemma 1.3. *Let us assume that* $\mathcal{F}[\phi(x)] = \Phi(w)$ *then we have the following Fourier transform identities*

$$1. \qquad \mathcal{F}[\phi(x-\beta); x \to w] = e^{i\beta w}\Phi(w).$$

$$2. \qquad \mathcal{F}[\int_a^x \phi(\xi)d\xi; x \to w] = \frac{1}{iw}\Phi(w).$$

Proof. See [4, 9].

Lemma 1.4. *Let us show that*

$$\mathcal{F}^{-1}[\int_{-\infty}^{+\infty} J_\nu(a\xi)\frac{e^{-|x|\sqrt{w^2+\lambda^2}}}{2\sqrt{\xi^2+\lambda^2}}\xi^{\nu+1}d\xi] = \sqrt{\frac{2}{\pi}}(\sqrt{x^2+\lambda^2})^\nu K_{\frac{\nu}{2}}(a\sqrt{x^2+\lambda^2}).$$

Note. In the above relation $J_\nu(.)$ and $K_\nu(.)$ are the Bessel function of the first kind of order ν and the modified Bessel function of the second kind of order ν respectively.
Proof. By definition of the inverse Fourier transform, the left hand side of the above relation can be rewritten as follows

$$L.H.S = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ixw}[\int_{-\infty}^{+\infty} J_\nu(a\xi)\frac{e^{-|w|\sqrt{w^2+\lambda^2}}}{2\sqrt{\xi^2+\lambda^2}}\xi^{\nu+1}d\xi]dw.$$

At this point, changing the order of integration yields

$$L.H.S = \int_{-\infty}^{+\infty} \frac{J_\nu(a\xi)}{2\sqrt{\xi^2+\lambda^2}}[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ixw}e^{-|w|\sqrt{w^2+\lambda^2}}dw]\xi^{\nu+1}d\xi,$$

in view of Lemma 1.2. the value of the inner integral is $\sqrt{\frac{2}{\pi}}\frac{\sqrt{\xi^2+\lambda^2}}{x^2+(\sqrt{\xi^2+\lambda^2})^2}$. After substitution and simplifying we arrive at

$$L.H.S = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\xi^{\nu+1}J_\nu(a\xi)}{\xi^2+(\sqrt{x^2+\lambda^2})^2}d\xi = \sqrt{\frac{2}{\pi}}\int_0^{+\infty} \frac{\xi^{\nu+1}J_\nu(a\xi)}{\xi^2+(\sqrt{x^2+\lambda^2})^2}d\xi$$

$$= \sqrt{\frac{2}{\pi}}(\sqrt{x^2+\lambda^2})^{\nu} K_{\frac{\nu}{2}}(2\sqrt{x^2+\lambda^2}).$$

□

The Fourier transform provides a useful technique for the solution of certain singular integral equations. Let us state and prove the following lemma.

Lemma 1.5. *Let us consider the following singular integral equation*

$$|x+\lambda|^{-\alpha} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\phi(\xi)}{|\xi - x|^{\beta}} d\xi, \qquad 0 < \alpha < \beta < 1, \lambda > 0.$$

Then the above singular integral equation has the formal solution as follows

$$\phi(x) = \sqrt{\frac{\pi}{2}} \frac{\cos(\frac{\pi\beta}{2})\Gamma(\beta)}{\cos(\frac{\pi\alpha}{2})(\beta-\alpha)\Gamma(\alpha)} \frac{|x+\lambda|^{\alpha-\beta-1}}{\Gamma(\beta-\alpha)}.$$

Proof. By applying the Fourier transform to each term in the integral equation and using the convolution Theorem and Lemma 1.2. the singular integral equation is converted into the following equation

$$\sqrt{\frac{2}{\pi}} \frac{\Gamma(1-\alpha)}{|w|^{1-\alpha}} e^{-i\lambda w} \sin(\frac{\pi\alpha}{2}) = \Phi(w) \sqrt{\frac{2}{\pi}} \frac{\Gamma(1-\beta)}{|w|^{1-\beta}} \sin(\frac{\pi\beta}{2}),$$

the above equation has the following solution

$$\Phi(w) = \frac{\Gamma(1-\alpha)}{\Gamma(1-\beta)|w|^{\beta-\alpha}} e^{-i\lambda w} \frac{\sin(\frac{\pi\alpha}{2})}{\sin(\frac{\pi\beta}{2})}.$$

At this point by applying the inverse Fourier transform, we obtain the formal solution as below

$$\phi(x) = \frac{\Gamma(1-\alpha)}{\Gamma(1-\beta)} \frac{\sin(\frac{\pi\alpha}{2})}{\sin(\frac{\pi\beta}{2})} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixw} |w|^{\alpha-\beta} e^{-i\lambda w} dw,$$

from which we deduce that

$$\phi(x) = \sqrt{\frac{\pi}{2}} \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\Gamma(\alpha)} \frac{\cos(\frac{\pi\beta}{2})}{(\beta-\alpha)\cos(\frac{\pi\alpha}{2})} |x+\lambda|^{\alpha-\beta-1}.$$

□

Let us consider the special case $\alpha = \frac{1}{3}, \beta = \frac{1}{2}, \lambda = 1$ we are led to the singular integral equation

$$|x+1|^{-\frac{1}{3}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\phi(\xi)}{|\xi - x|^{\frac{1}{2}}} d\xi,$$

the solution of which is given by

$$\phi(x) = \frac{\pi\sqrt{6}}{\Gamma(\frac{1}{6})} |x+1|^{-\frac{7}{6}}.$$

Lemma 1.6. *Let us consider the following integral equation with retarded argument*

$$\phi(x-\beta) = f(x) + \lambda \int_a^x \phi(\xi)d\xi, \qquad \lambda > 0.$$

Then the above integral equation has the formal solution as follows

$$\phi(x) = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1} f^{(k)}(x-k\beta)}{\lambda^{k+1}}.$$

Proof. Taking the Fourier transform of the above integral equation term-wise and in view of Lemma1.3. We get

$$e^{i\beta w}\Phi(w) = F(w) + \frac{\lambda}{iw}\Phi(w),$$

from which we deduce that

$$\Phi(w) = \frac{F(w)}{e^{i\beta w} - \frac{\lambda}{iw}} = -\frac{1}{\lambda}\sum_{k=0}^{+\infty}(\frac{iwe^{iw\beta}}{\lambda})^k F(w) = -\sum_{k=0}^{+\infty}\frac{(-1)^k}{\lambda^{k+1}}(-iw)^k e^{ik\beta w}F(w).$$

By taking the inverse Fourier transform we obtain

$$\phi(x) = -\sum_{k=0}^{+\infty}\frac{(-1)^k}{\lambda^{k+1}}[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i(x-k\beta)w}(-iw)^k F(w)dw.]$$

At this point using the fact that $f^{(k)}(\eta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(-iw)^k e^{-i\eta w}F(w)dw$, we have

$$\phi(x) = \sum_{k=0}^{+\infty}\frac{(-1)^{k+1}}{\lambda^{k+1}} f^{(k)}(x-k\beta)$$

□

Definition 1.2. The left Caputo fractional derivative of order α $(0<\alpha<1)$ of $\phi(t)$ is defined as follows [12]

$$D_{a,t}^{c,\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{1}{(t-\xi)^\alpha}\phi'(\xi)d\xi. \tag{1.3}$$

Definition 1.3. Laplace transform of the function $\phi(t)$ is defined as follows

$$\mathcal{L}\{\phi(t)\} = \int_0^\infty e^{-st}\phi(t)dt := \Phi(s). \tag{1.4}$$

The sufficient conditions for the existence of the Laplace transform are that the function $\phi(t)$ be of exponential order and be sectionally continous on every closed interval

$0 \le t \le \lambda$ for every positive λ. The inverse Laplace transform of $\Phi(s)$ may be expressed explicitly as a contour integral by considering s as a complex variable. If $\mathcal{L}\{\phi(t)\} = \Phi(s)$, then $\mathcal{L}^{-1}\{\Phi(s)\}$ is given by

$$\phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\Phi(s)ds, \tag{1.5}$$

where $\Phi(s)$ is analytic in the region $\mathrm{R}e(s) > c$.
Note. A theorem due to Lerch states that if two functions have the same Laplace transform they defer by a null function $\mathcal{N}(t)$, which has the property that, for every $\lambda > 0$,

$$\int_0^\lambda \mathcal{N}(t)dt = 0.$$

Definition 1.4. The Laplace transform of the Caputo fractional derivatives of order non-integer α. The most important use of the Caputo fractional derivative is treated in initial value problems where the initial conditions are expressed in terms of integer order derivatives. In this respect, it is interesting to know the Laplace transform of this kind of derivative.

$$\mathcal{L}\{D_{0,t}^{c,\alpha} f(t)\} = sF(s) - f(0+), 0 < \alpha < 1. \tag{1.6}$$

and generally [12]

$$\mathcal{L}\{D_{0,t}^{c,\alpha} f(t)\} = s^{\alpha-1}F(s) - \textstyle\sum_{k=0}^{k=m-1-k} s^{\alpha-1-k} f^k(0+), m-1 < \alpha < m. \tag{1.7}$$

Example 1.1. Using convolution theorem for the Laplace transform to show that

$$\int_0^t \frac{J_\nu(\alpha\xi)J_\lambda(\alpha(t-\xi))}{\xi(t-\xi)}d\xi = (\frac{1}{\nu} + \frac{1}{\lambda})\frac{J_{\nu+\lambda}(\alpha t)}{t}, \quad \nu, \lambda > 0.$$

Where $J_\nu(.)$ stands for the Bessel function of the first kind of order ν.
Solution. From table of the Laplace transforms it is well known that [7]

$$\mathcal{L}[\frac{J_\nu(at)}{t}] = \frac{(\sqrt{s^2+a^2} - s)^\nu}{\nu a^\nu}.$$

In view of the convolution Theorem for the Laplace transform we have the following

$$\mathcal{L}[\int_0^t \frac{J_\nu(\alpha\xi)J_\lambda(\alpha(t-\xi))}{\xi(t-\xi)}d\xi] = \mathcal{L}[\frac{J_\nu(at)}{t}]\mathcal{L}[\frac{J_\lambda(at)}{t}] = \frac{(\sqrt{s^2+a^2} - s)^\nu}{\nu a^\nu}\frac{(\sqrt{s^2+a^2} - s)^\lambda}{\lambda a^\lambda},$$

after simplifying we arrive at

$$\mathcal{L}[\int_0^t \frac{J_\nu(\alpha\xi)J_\lambda(\alpha(t-\xi))}{\xi(t-\xi)}d\xi] = \frac{(\sqrt{s^2+a^2} - s)^{\nu+\lambda}}{\nu\lambda a^{\nu+\lambda}} = (\frac{1}{\nu} + \frac{1}{\lambda})\frac{(\sqrt{s^2+a^2} - s)^{\nu+\lambda}}{(\nu+\lambda)a^{\nu+\lambda}}.$$

At this stage taking the inverse Laplace transform of the above relation, we obtain

$$\int_0^t \frac{J_\nu(\alpha\xi)J_\lambda(\alpha(t-\xi))}{\xi(t-\xi)}d\xi = \mathcal{L}^{-1}[(\frac{1}{\nu} + \frac{1}{\lambda})\frac{(\sqrt{s^2+a^2} - s)^{\nu+\lambda}}{(\nu+\lambda)a^{\nu+\lambda}}] = (\frac{1}{\nu} + \frac{1}{\lambda})\frac{J_{\nu+\lambda}(at)}{t}.$$

Example 1.2. Using Bromwich complex inversion formula and residue theorem to show that

$$\begin{aligned}\phi(t) &= \mathcal{L}^{-1}[\frac{s^m K_1(\alpha\sqrt{s})}{\sqrt{s}-\lambda}] \\ &= 2\lambda^{2m+1}K_1(a\lambda)e^{\lambda^2 t} - \frac{1}{2}\int_0^{+\infty}(-r)^m e^{-tr}\frac{\lambda J_1(a\sqrt{r})+\sqrt{r}Y_1(a\sqrt{r})}{\lambda^2+r}dr.\end{aligned}$$

$$\lambda > 0, \quad m = 0, 1, 2, 3, ...$$

Where $K_\nu(.)$ stands for the modified Bessel function of the second kind of order ν or Macdonald's function [1].

Solution. The transform function $\Phi(s) = \frac{s^m K_1(\alpha\sqrt{s})}{\sqrt{s}-\lambda}$ has a simple pole at $s = \lambda^2$ and branch point at $s = 0$. Then the inverse Laplace transform is

$$\phi(t) = \lim_{s\to\lambda^2}[(s-\lambda^2)\frac{s^m K_1(\alpha\sqrt{s})e^{st}}{\sqrt{s}-\lambda}]$$

$$+\frac{1}{\pi}\int_0^{+\infty} e^{-tr} Im[\lim_{\theta\to-\pi}\Phi(re^{i\theta})]dr,$$

let us evaluate each term as follows

$$\phi(t) = \lim_{s\to\lambda^2}(\sqrt{s}-\lambda)(\sqrt{s}+\lambda).\frac{s^m K_1(a\sqrt{s})e^{st}}{\sqrt{s}-\lambda}$$

$$+\frac{1}{\pi}\int_0^{+\infty} e^{-tr} Im[\lim_{\theta\to-\pi}\frac{(re^{i\theta})^m K_1(\alpha\sqrt{re^{i\theta}})}{\sqrt{re^{i\theta}}-\lambda}]dr.$$

After evaluation of the limits and simplifying we get

$$\phi(t) = 2\lambda^{2m+1}K_1(a\lambda)e^{\lambda^2 t} - \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\Im[\frac{(-r)^m K_1(-ia\sqrt{r})}{\lambda+i\sqrt{r}}]dr,$$

or

$$\phi(t) = 2\lambda^{2m+1}K_1(a\lambda)e^{\lambda^2 t} - \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\Im[\frac{(-r)^m(\lambda-i\sqrt{r})K_1(-ia\sqrt{r})}{\lambda^2+r}]dr.$$

At this point let us recall the following well-known identity for the Bessel's function

$$K_\nu(z) = \frac{i\pi}{2}e^{\frac{1\pi\nu}{2}}[J_\nu(e^{\frac{i\pi}{2}}z) + iY_\nu(e^{\frac{i\pi}{2}}z)].$$

Thus, we get

$$\phi(t) = 2\lambda^{2m+1}K_1(a\lambda)e^{\lambda^2 t} - \frac{1}{2}\int_0^{+\infty} e^{-tr}\Im[\frac{(-r)^m(i\lambda+\sqrt{r})(J_1(a\sqrt{r})+iY_1(a\sqrt{r}))}{\lambda^2+r}]dr.$$

After simplification we obtain

$$\phi(t) = 2\lambda^{2m+1}K_1(a\lambda)e^{\lambda^2 t} - \frac{1}{2}\int_0^{+\infty}(-r)^m e^{-tr}\frac{\lambda J_1(a\sqrt{r})+\sqrt{r}Y_1(a\sqrt{r})}{\lambda^2+r}dr.$$

Definition 1.5. The generalized Laguerre polynomials $L_n^\alpha(x)$ satisfy the linear differential equation with non-constant coefficients

$$xy'' + (\alpha + 1 - x)y + ny = 0.$$

The generating function is

$$(1-t)^{-(\alpha+1)} \exp(-\frac{tx}{1-t}) = \sum_{n=0}^{+\infty} L_n^\alpha(x) t^n, \quad |t| < 1.$$

upon comparing coefficients of t^n in the two series expasions of the generating function, we obtain

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n(-x)^k}{k!(n-k)!(1+\alpha)_k}.$$

In special case $\alpha = 0$, we have

$$L_n^0(x) = L_n(x) = \sum_{k=0}^{n} \frac{n!(-x)^k}{(k!)^2(n-k)!} = \sum_{k=0}^{k=n} C_k^n \frac{(-x)^k}{k!},$$

from which we deduce that

$$\mathcal{L}[L_n(x); x \to s] = \frac{1}{s}(1 - \frac{1}{s})^n.$$

Note. The most important application of the Laguerre polynomials is in the quantum-mechanical analysis of the hydrogen atom.

Corollary 1.1. *The following identities hold true*

$$1. \qquad e^t J_0(2\sqrt{tx}) = \sum_{n=0}^{+\infty} \frac{L_n(x)t^n}{n!}.$$

$$2. \qquad e^{-t} I_0(2\sqrt{tx}) = \sum_{n=0}^{+\infty} \frac{(-1)^n L_n(x)t^n}{n!}.$$

Proof. Part(1). In view of the Lerch's theorem, by taking the Laplace transform of both sides with respect to $x, x > 0$ after some manipulations we get the same result. Part(2). In part (1), let us change t to $-t$ and using the fact that $J_0(it) = I_0(t)$ we get

$$e^{-t} J_0(2\sqrt{-tx}) = e^{-t} I_0(2\sqrt{tx}) = \sum_{n=0}^{+\infty} \frac{L_n(x)(-t)^n}{n!} = \sum_{n=0}^{+\infty} (-1)^n \frac{L_n(x)t^n}{n!}.$$

$\square$

Definition 1.6. We define the associated Laguerre transform of the function $\phi(x)$ as follows

$$L_{n,\alpha}[\phi(x)] = \Phi_L(n,\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha} L_n^{\alpha}(x)\phi(x)dx,$$

and the inverse transform

$$L_{n,\alpha}^{-1}[\Phi_L(n,\alpha)] = \phi(x) = \sum_{n=0}^{+\infty} \frac{n!}{\Gamma(n+\alpha+1)} L_n^{\alpha}(x)\Phi_L(n,\alpha).$$

In special case $\alpha = 0$, we define the Laguerre transform of the function $\phi(x)$ as follows

$$L_n[\phi(x)] = \Phi_L(n) = \int_0^{+\infty} e^{-x} L_n(x)\phi(x)dx,$$

and the inverse transform

$$L_n^{-1}[\Phi_L(n)] = \phi(x) = \sum_{n=0}^{+\infty} L_n(x)\Phi_L(n).$$

Remark 1.1. Let us consider the generalized Laguerre differential equation in self-adjoint form,

$$[(x^{\alpha+1}e^{-x})y']' + nx^{\alpha}e^{-x}y = 0,$$

we note that

$$L_{n,\alpha}[xy'' + (\alpha+1-x)y'] = -nL_{n,\alpha}y.$$

Thus, the Laguerre transform is suited for application to partial differential equations containing terms of the type

$$\mathcal{M}\psi = x\frac{\partial^2\psi}{\partial x^2} + (\alpha+1-x)\frac{\partial\psi}{\partial x}.$$

In special case $\alpha = 0$ we have

$$\mathcal{M}\psi = x\frac{\partial^2\psi}{\partial x^2} + (1-x)\frac{\partial\psi}{\partial x}.$$

Lemma 1.7. *The following identities hold true*

$$\sum_{n=0}^{+\infty} L_n(\xi)L_n(x)\theta^n = \frac{1}{1-\theta} e^{-\frac{\theta(x+\xi)}{1-\theta}} I_0(\frac{2\sqrt{\theta x\xi}}{1-\theta}).$$

$$\sum_{n=0}^{+\infty} L_n(\xi)L_n(x) = e^{\xi}\delta(x-\xi).$$

Proof. See [6].

Corollary 1.2. *In the first part of the above Lemma, if we set $\theta = e^{-t}$ we get the following result*

$$\sum_{n=0}^{+\infty} L_n(\xi)L_n(x)e^{-nt} = \frac{1}{1-e^{-t}} e^{-\frac{e^{-t}(x+\xi)}{1-e^{-t}}} I_0(\frac{2\sqrt{e^{-t}x\xi}}{1-e^{-t}}).$$

2. Main Result (Exact solution to non-homogenous time fractional PDE via the Joint Laplace-Laguerre transforms)

The fractional derivatives are powerful technique for solving differential equations resulted from several physical modeling such as the fractional diffusion-wave equation, for more details see Mainardi [10, 11], Das [8]. However, some other researchers worked on the existence and uniqueness of solutions to some differential equations with fractional order (see Podlubny [12]).
In [2, 5] the author has used operational method to find analytical solutions of certain partial fractional differential equations. In this section, the author implemented the joint Laplace-Laguerre transform to construct exact solution for a variant of the time fractional heat conduction equation with non-constant coefficients and non-homogeneous initial condition.

Problem 1. Let us consider the following time fractional PDE with non-constant coefficints

$$D_t^{C,\alpha} u(x,t) = x u_{xx} + (\lambda + 1 - x) u_x, \quad 0 < \alpha < 1, \quad x, t > 0.$$

$$u(x,0) = g(x), \quad |u(x,t)| < e^{kx}, k > 1, x \to +\infty.$$

Solution. Let us define the joint Laplace-Laguerre transforms of the function $u(x,t)$ as follows

$$\mathcal{L}[L_{n,\lambda} u(x,t); x \to n]; t \to s] = U_{n,\lambda}(n,s) = \int_0^{+\infty} e^{-st} [\int_0^{+\infty} x^\lambda e^{-x} L_n^\lambda(x) u(x,t) dx] dt.$$

Note. It is worth mentioning that the joint Laplace-Laguerre transforms is very similar to the two dimensional Laplace transforms [6].
We find that the joint transforms applied to the problem leads to the transformed equation as below

$$s^\alpha U_{L,\lambda}(n,s) - s^{\alpha-1} G_{L,\lambda}(n) = -n U_{L,\lambda}(n,s),$$

where

$$U_{L,\lambda}(n,0) = G_{L,\lambda}(n) = \int_0^{+\infty} e^{-\xi} \xi^\lambda L_n^\lambda(\xi) g(\xi) d\xi.$$

From which we deduce that

$$U_{L,\lambda}(n,s) = \frac{s^{\alpha-1} G_{L,\lambda}(n)}{s^\alpha + n}.$$

At this stage, taking the inverse joint Laplace-Lageurre transforms we obtain

$$u(x,t) = \sum_{n=0}^{+\infty} \frac{n! L_n^\lambda(x) G_{L,\lambda}(n)}{\Gamma(\lambda + n + 1)} [\mathcal{L}^{-1}(\frac{s^{\alpha-1}}{s^\alpha + n})].$$

Let us recall the Laplace transform of the pair of functions [12],

$$\int_0^{+\infty} e^{-st}E_\alpha(-nt^\alpha)dt = \frac{s^{\alpha-1}}{s^\alpha+n}, \quad E_\alpha(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\alpha n+1)}.$$

Note. $E_\alpha(z)$ stands for the Mittag-Leffler function. The Mittag-Leffler function is the basis function of the fractional calculus. Several modifications of the Mittag-Leffler functions are introduced for study of the fractional calculus [12].
Thus, we have

$$u(x,t)=\sum_{n=0}^{+\infty}\frac{n!L_n^\lambda(x)G_{L,\lambda}(n)}{\Gamma(\lambda+n+1)}E_\alpha(-nt^\alpha),$$

and so we finally deduce that

$$u(x,t)=\int_0^{+\infty} e^{-\xi}\xi^\lambda g(\xi)[\sum_{n=0}^{+\infty}\frac{n!L_n^\lambda(x)L_n^\lambda(\xi)}{\Gamma(\lambda+n+1)}E_\alpha(-nt^\alpha)]d\xi.$$

Let us study the following special cases
1. $\lambda=0,\ \alpha=1,\ g(x)=\delta(x-\beta)$, we get

$$u(x,t)=\int_0^{+\infty} e^{-\xi}g(\xi)[\sum_{n=0}^{+\infty}L_n(x)L_n(\xi)e^{-nt}]d\xi = e^{-\beta}[\sum_{n=0}^{+\infty}L_n(x)L_n(\beta)e^{-nt}].$$

After using the above Corollary 1.2. we get finally

$$u(x,t)=\frac{e^{-\beta}}{1-e^{-t}}e^{-\frac{e^{-t}(x+\beta)}{1-e^{-t}}}I_0(\frac{2\sqrt{e^{-t}x\beta}}{1-e^{-t}}).$$

Note. It is easy to verify that $u(x,0)=\delta(x-\beta)$, in view of Lemma 1.5. we have

$$u(x,0)=\int_0^{+\infty} e^{-\xi}\delta(\xi-\beta)[\sum_{n=0}^{+\infty}L_n(x)L_n(\beta)]d\xi = e^{-\beta}[\sum_{n=0}^{+\infty}L_n(x)L_n(\xi)]$$
$$= e^{-\beta}[e^{\beta}\delta(x-\beta)]=\delta(x-\beta).$$

2. For $\alpha=0.5,\ \lambda=0$, we have

$$u(x,t)=\sum_{n=0}^{+\infty}\frac{n!L_n(x)G_L(n)}{\Gamma(n+1)}[\mathcal{L}^{-1}(\frac{1}{\sqrt{s}(\sqrt{s}+n)})]=\sum_{n=0}^{+\infty}L_n(x)G_L(n)e^{n^2t}Erfc(n\sqrt{t}),$$

or

$$u(x,t)=\sum_{n=0}^{+\infty}L_n(x)G_L(n)e^{n^2t}Erfc(n\sqrt{t})$$
$$=\sum_{n=0}^{+\infty}L_n(x)e^{n^2t}Erfc(n\sqrt{t})[\int_0^{+\infty} e^{-\xi}g(\xi)L_n(\xi)d\xi].$$

Interchanging the order of the summation and integration, we arrive at

$$u(x,t) = \int_0^{+\infty} e^{-\xi} g(\xi)[\sum_{n=0}^{+\infty} L_n(x) L_n(\xi) e^{n^2 t} Erfc(n\sqrt{t})] d\xi,$$

in view of Lemma 1.5. we have

$$\begin{aligned} u(x,0) &= \int_0^{+\infty} e^{-\xi} g(\xi)[\sum_{n=0}^{+\infty} L_n(x) L_n(\xi) Erfc(n\sqrt{0})] d\xi \\ &= \int_0^{+\infty} e^{-\xi} g(\xi) e^{\xi} \delta(\xi - x) = g(x). \end{aligned}$$

3. Conclusion

The article is devoted to study and applications of the Fourier, Laplace and Laguerre transforms for solving certain singular integral equation, integral equation with retarded argument, and time fractional heat equation wth non-constant coefficients. The properties included in this article indicate the take-off points for advanced and modern developments in this field.

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