On parameters of independence, domination and irredundance in edge-coloured graphs and their products

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Abstract: In this paper we study some parameters of domination, independence and irredundance in some edge-coloured graphs and their products. We present several general properties of independent, dominating and irredundance sets in edge-coloured graphs and we give relationships between the independence, domination and irredundant numbers of an edge-coloured graph. We generalize some classical results concerning independence, domination and irredundance in graphs. Moreover we study $G$-join of edge-coloured graphs which preserves considered parameters with respect to related parameters in product factors.

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1 Introduction

Consider a finite connected graph $G$ with a vertex set $V(G)$ and an edge set $E(G)$. A path from a vertex $x_1$ to a vertex $x_n$, $n \geq 2$, in $G$ is a sequence of distinct vertices $x_1, \ldots, x_n$ such that $x_i x_{i+1} \in E(G)$, for $i = 1, \ldots, n-1$; we denote it simply by $x_1 \cdots x_n$. If $x_1 = x_n$ the path form a cycle. An edge-$m$-colouring of $G$ is a mapping $c : E(G) \to \{1, \ldots, m\}$. We then say that $G$ is edge-$m$-coloured by $c$. An $m$-coloured graph $G$ is monochromatic if $c(e) = c(f)$ for any $e, f \in E(G)$. We abuse the notation slightly and call $c(G) = c(e)$, for any $e \in E(G)$. A path (cycle) is $m$-coloured if its edges are coloured using $m$-colours. A path is called monochromatic if its edges are coloured alike. The set of all vertices $y$ for which there is a monochromatic path $y \cdots x$ is called the chromatic neighborhood of $x$ and is denoted by $N_{mp}^G(x)$. We write $N_{G}^{mp}$.\[\text{COPYRIGHT â© by Publishing Department Rzeszów University of Technology}
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Instead of $N_G^{mp}(x) \cup \{x\}$. For a subset $X$ of $V(G)$ we write $N_G^{mp}(X)$ and $N_G^{mp}[X]$ instead of $\bigcup_{x \in X} N_G^{mp}(x)$ and $\bigcup_{x \in X} N_G^{mp}[X]$, respectively.

A subset $S \subset V(G)$ is said to be independent by monochromatic paths of the edge-coloured graph $G$ if for any two different vertices $x, y \in S$ there is no monochromatic path between them. In addition a subset containing only one vertex and the empty set are independent by monochromatic paths. For convenience we will write an imp-set of $G$ instead of an independent by monochromatic paths set of $G$. For any proper edge-colouring of the graph $G$ an imp-set of $G$ is an independent set in the classical sense. Moreover every imp-set of $G$ is independent. The lower and upper independence by monochromatic paths numbers $i_{mp}(G)$ and $\alpha_{mp}(G)$ of $G$ are respectively the minimum and maximum cardinalities of maximal imp-set of vertices of $G$.

A subset $Q \subseteq V(G)$ is dominating by monochromatic paths, shortly dmp-set of the edge-coloured graph $G$ if for each $x \in V(G) \setminus Q$ there exists a monochromatic path $x \ldots y$, for some $y \in Q$. We will write a dmp-set of $G$ instead of dominating by monochromatic paths set of $G$. For proper edge-colouring of the graph $G$ a dmp-set of $G$ is a dominating set of $G$ in the classical sense. Moreover every dominating set of $G$ is a dmp-set. The lower and upper by monochromatic paths numbers $\gamma_{mp}(G)$ and $\Gamma_{mp}(G)$ of $G$ are respectively the minimum and maximum cardinalities of minimal dmp-set of vertices of $G$.

Parameters $\gamma_{mp}(G)$ and $\alpha_{mp}(G)$ will be named as the domination by monochromatic paths and independence by monochromatic paths numbers, respectively.

Let $G$ be an edge-coloured graph and $X \subset V(G)$. For every $x \in X$, define $I_G^{mp}(x, X) = N_G^{mp}[x] - N_G^{mp}[X - \{x\}]$ the set of private chromatic neighbours of the vertex relative to the set $X$. If $I_G^{mp}(x, X) = \emptyset$, then $x$ is said to be redundant by monochromatic path in $X$. A set $X$ of vertices containing no redundant by monochromatic paths vertex is called irredundant by monochromatic paths. The lower and upper irredundancy by monochromatic paths number $ir_{mp}(G)$ and $IR_{mp}(G)$ of a graph $G$ are respectively the minimum and maximum cardinalities of maximal irredundant by monochromatic paths set of vertices of $G$. The parameter $ir_{mp}(G)$ is the irredundance by monochromatic paths number of an edge-coloured graph $G$. In this paper we will write an irmp-set of $G$ instead of an irredundant by monochromatic paths set of $G$. For the proper edge-colouring of the graph $G$ an irmp-set of $G$ is an irredundant set in the classical sense.

Note that for the proper edge-colouring of the graph $G$ we have the following equalities: $\alpha_{mp}(G) = \alpha(G)$, $\gamma_{mp}(G) = \gamma(G)$, $\Gamma_{mp}(G) = \Gamma(G)$, $i_{mp}(G) = i(G)$, $ir_{mp}(G) = ir(G)$ and $IR_{mp}(G) = IR(G)$.

The concepts of independence, domination and irredundance have existed in literature for a long time, see [14]. There are several generalizations of these concepts, for instance generalization in distance sense see [10, 13].

Concept of independence and domination by monochromatic path in graphs were studied in [1-8] and [15-21]. More generalized concept was considered recently in [9].

In this paper we study parameters of independence, domination and irredundance by monochromatic paths in an edge-coloured graphs and their products. We give some general properties of imp-sets, dmp-sets and irmp-sets in an edge-coloured graph and
we give some relationships between studied parameters which generalize results for independent sets, dominating sets and irredundance sets in the classical sense.

2 General properties of imp-sets and dmp-sets in graphs

In this section we give some relations between imp-sets, dmp-sets and irmp-sets in graphs.

**Theorem 2.1.** [20] For an arbitrary edge-coloured graph $G$ and a subset $S \subset V(G)$ the following conditions are equivalent:

1. $S$ is a maximal imp-set of $G$.
2. $S$ is an imp-set of $G$ and a dmp-set of $G$.
3. $S$ is both a maximal imp-set and a minimal dmp-set of $G$.

**Theorem 2.2.** Let $X$ be an irmp-set of an edge-coloured graph $G$. If there exists $x \in X$ such that $I_{mp}^G(x, X) \subseteq N_{mp}^G[v]$ for some $v \in V(G) - N_{mp}^G[X]$, then $I_{mp}^G(x, X) \neq \emptyset$, for any $v \in V(G) - N_{mp}^G[X]$.

**Proof.** Assume that there exists $x \in X$ such that $I_{mp}^G(x, X) \subseteq N_{mp}^G[v]$ for some $v \in V(G) - N_{mp}^G[X]$. Then $x \in I_{mp}^G(x, X) \subseteq N_{mp}^G[v]$, that is, $v \in N_{mp}^G[x]$, which contradict the choice of the vertex $v \in V(G) - N_{mp}^G[X]$.

**Theorem 2.3.** Let $G$ be an edge-coloured graph and let $Q$ be a dmp-set in $G$. Then $Q$ is a minimal dmp-set in $G$ if and only if $I_{mp}^G(x, Q) \neq \emptyset$, for each $x \in Q$.

**Proof.** If $Q$ is a minimal dmp-set in $G$, then for each $x \in Q$ we have that $N_{mp}^G[x] \cup N_{mp}^G[Q - \{x\}] = N_{mp}^G[Q] = V(G)$. Since $N_{mp}^G[Q - \{x\}] \subset V(G)$, so $I_{mp}^G(x, Q) \neq \emptyset$. Assume now that $Q$ is a dmp-set in $G$ and $I_{mp}^G(x, Q) \neq \emptyset$, for each $x \in Q$. Suppose on contrary that $Q$ is not minimal. This means that for some $x \in Q$, $Q - \{x\}$ is a dmp-set in $G$. Therefore $N_{mp}^G[Q - \{x\}] = V(G)$ and since $N_{mp}^G[x] \subset V(G)$, so $I_{mp}^G(x, Q) = \emptyset$, contrary to the hypothesis.

From the definition of an irmp-set and Theorem 2.3 it follows the following relationships between minimal dmp-sets and maximal irmp-sets:

**Corollary 1.** Let $Q$ be a dmp-set of an edge-coloured graph $G$. Then $Q$ is a minimal dmp-set of $G$ if and only if $Q$ is a maximal irmp-set of $G$.

In view of the facts that every maximal imp-set of a graph $G$ is a minimal dmp-set and every minimal dmp-set is a maximal irmp-set it follows the following string of inequalities:

**Proposition 2.4.** For any edge-coloured graph $G$,

$$ir_{mp}(G) \leq \gamma_{mp}(G) \leq i_{mp}(G) \leq \alpha_{mp}(G) \leq \Gamma_{mp}(G) \leq IR_{mp}(G).$$
Theorem 2.5. If $X$ is a smallest maximal irmp-set in an edge-coloured graph $G$ and $X$ is an imp-set, then

$$ir_{mp}(G) = \gamma_{mp}(G) = i_{mp}(G).$$

Proof. By Proposition 2.4 we obtain that $ir_{mp}(G) \leq \gamma_{mp}(G) \leq i_{mp}(G)$, so it suffices to prove that $ir_{mp}(G) = i_{mp}(G)$. Suppose on the contrary that $ir_{mp}(G) \neq i_{mp}(G)$. Then $|X| = ir_{mp}(G) < i_{mp}(G)$ and this implies that $X$ is not a maximal imp-set in $G$. Consequently $V(G) - N_{G}[X] \neq \emptyset$ and for any $x \in V(G) - N_{G}[X]$ the set $X \cup \{x\}$ is an imp-set of $G$. Therefore by previous considerations $X \cup \{x\}$ is an imp-set in $G$, contrary to the maximality of $X$. \qed

Theorem 2.6. Let $G_1, ..., G_n$ be the connected components of an edge-coloured graph $G$ and $X$ be a maximal irmp-set of $G$ and $X_i = X \cap V(G_i)$. Then $X_i \neq \emptyset$ and $X_i$ is a maximal irmp-set of $G_i$, for each $i = 1, ..., n$.

Proof. If $X_i = \emptyset$ for some $i$, $1 \leq i \leq n$ then we can observe that $X \cup \{y\}$ is an irmp-set of $G$ for any $y \in V(G_i)$ which contradicts the maximality of $X$. This implies that $X_i \neq \emptyset$ for each $i = 1, ..., n$. Because $X$ is a maximal irmp-set of $G$ hence $I_{mp}(x, X) \neq \emptyset$, for each $x \in X$ and since $X_i \subseteq X$, so $X_i$ is also an irmp-set of $G_i$. Suppose that there exists $1 \leq i \leq n$ such that $X_i$ is not a maximal irmp-set of $G_i$. Then there exists at least one vertex $y \in V(G_i)$ such that $X_i \cup \{y\}$ is also an irmp-set, and consequently we have that $X \cup \{y\}$ is also an irmp-set of $G$, a contradiction to the maximality of $X$. \qed

Theorem 2.7. Let $G$ be an edge-coloured graph. If $X$ is a maximal irmp-set of $G$ then for any $u \in V(G) - N_{G}[X]$ there exists some $x \in X$ such that

1. $I_{mp}^G(x, X) \subseteq N_{G}^G(u),$
2. for $x_1, x_2 \in I_{mp}^G(x, X)$ such that $x_1 \neq x_2$ either there is a monochromatic path $x_1 ... x_2$ in $G$ or there exist $y_1, y_2 \in X - \{x\}$ such that there is a monochromatic path from $x_1$ to each vertex of $I_{mp}^G(y_1, X)$ and there is a monochromatic path from $x_2$ to each vertex of $I_{mp}^G(y_2, X)$.

Proof. (1). From the assumption about maximality of $X$ we obtain that the set $X \cup \{u\}$ is not an irmp-set in $G$. Consequently $I_{mp}^G(x, X \cup \{u\}) = \emptyset$ for some $x \in X \cup \{x\}$. Since $u \in V(G) - N_{G}[X]$, hence there is no monochromatic path $u ... y$ for every $y \in X$, so $u \in I_{mp}^G(u, X \cup \{u\})$ and therefore $x \neq u$. Because $I_{mp}^G(x, X \cup \{u\}) = N_{G}[x] - N_{G}[X \cup \{u\}] = N_{G}[x] - N_{G}[X - \{x\}] = N_{G}[x] - N_{G}^G[u] = \emptyset$, then $I_{mp}^G(x, X) \subseteq N_{G}[x] - N_{G}[X - \{x\}] \subseteq N_{G}[u]$ and this gives $I_{mp}^G(x, X) \subseteq N_{G}^G(u)$ as $u \notin I_{mp}^G(x, X)$.

(2). Let $x_1, x_2$ be two distinct vertices of $I_{mp}^G(x, X)$ such that there is no monochromatic path $x_1 ... x_2$ in $G$ and suppose on the contrary that for $x_1$ or $x_2$ say for $x_1$ and for all $y \in X - \{x\}$ there is $z_i \in I_{mp}^G(y, X)$ that there are no monochromatic paths $z_i ... x_1$ in $G$. Then $x_2 \in I_{mp}^G(x, X \cup \{z_1\})$, $u \in I_{mp}^G(x_1, X \cup \{x_2\})$, $z_i \in I_{mp}^G(y, X \cup \{x_1\})$ for each $y \in X - \{x\}$ and therefore $X \cup \{x_1\}$ is an irmp-set in $G$, which contradicts the maximality of $X$. \qed
Theorem 2.8. Let $X$ be a smallest maximal $irmp$-set in $G$. Let $X_0 \subset X$ be a subset such that $X_0 \cup \{x\}$ is an imp-set in $G$, for each $x \in X$ and $|X_0| = k < |X|$. Then $\gamma_{mp}(G) \leq 2\ir_{mp}(G) - k - 1$

Proof. Because $|X_0| = k < |X|$ so $X - X_0 \neq \emptyset$. Let $X - X_0 = \{x_1, \ldots, x_n\}$. Clearly $n \geq 2$. For each $x_i \in X - X_0$ we choose any $x_i' \in I_{G}^{mp}(x_i, X)$ and we define the set $X' = X \cup \{x_1', \ldots, x_n'\}$. From the assumption of $X_0$ we obtain that $x_i \notin I_{G}^{mp}(x_i, X)$, so $x_i' \neq x_i$ for $i = 1, \ldots, n$ and therefore $|X'| \leq 2\ir_{mp}(G) - k$. Moreover the assumption of the set $X_0$ implies that for every $x_p \in X - X_0$ there is $x_q \in X - X_0$ and a monochromatic path $x_p \ldots x_q$ in $G$. We shall show that $X'$ is a dmp-set in $G$. Assume on the contrary that $X'$ is not a dmp-set and let $u \in V(G) - N_{G}^{mp}[X']$. Consequently $X$ is not a dmp-set in $G$. Thus for every $y \in X$ there is no monochromatic path $u \ldots y$ in $G$. Then Theorem 2.7 (1) gives that $I_{G}^{mp}(x, X) \subseteq N_{G}^{mp} (u)$ for some $x \in X$. If $x \in X_0$, then $x \in I_{G}^{mp}(x, X)$ and there is a monochromatic path $x \ldots u$, which contradicts the assumption. If $x \in X - X_0$, then $x = x_i$ (for some $i \in \{1, \ldots, n\}$) and by previous considerations $x_i \in I_{G}^{mp}(x_i, X)$. Because $I_{G}^{mp}(x_i, X) \subseteq N_{G}^{mp} (x_i)$ so there is a monochromatic path $x_i' \ldots u$ in $G$, a contradiction. Therefore $X'$ is a dmp-set. Since $X \subset X'$, hence Corollary 1 implies that $X'$ is not a minimal dmp-set. Consequently $\gamma_{mp}(G) < |X'| \leq 2\ir_{mp}(G) - k$ and $\gamma_{mp} \leq 2\ir_{mp}(G) - k - 1$.

Corollary 2. For any edge-coloured graph $G$, $\gamma_{mp}(G) \leq 2\ir_{mp}(G) - 1$.

Proof. Let $X$ be a smallest maximal $irmp$-set in $G$. If $X$ is an imp-set then by Proposition 2.4 we have that $\gamma_{mp}(G) \leq \ir_{mp}(G)$ and therefore $\gamma_{mp}(G) \leq 2\ir_{mp}(G) - 1$. If $X$ is not an imp-set and $G[X]$ has a subset $X_0$ on $k$ vertices such that $X_0 \cup \{x\}$ is an imp-set in $G$, for each $x \in X$, then by Theorem 2.8 it follows that $\gamma_{mp}(G) \leq 2\ir_{mp}(G) - k - 1 \leq 2\ir_{mp}(G) - 1$.

A vertex $x$ of an edge-coloured graph is called a monochromatic vertex if it belongs to exactly one maximal (with respect to set inclusion) connected monochromatic subgraph of $G$. A connected monochromatic subgraph of a graph $G$ containing at least one monochromatic vertex is called a monochromatic simplex of $G$. Note that if $x$ is a monochromatic vertex of $G$ then $G[N_{G}[x]]$ contains a subgraph being the unique monochromatic simplex of $G$ containing $x$. A graph $G$ is monochromatic simplical if every vertex of $G$ is a monochromatic vertex or belong to the monochromatic simplex. Certainly, if $G$ is a monochromatic simplical graph and $M_1, \ldots, M_n$ are the monochromatic simplices in $G$, then $V(G) = \bigcup_{i=1}^{n} V(M_i)$.

The following theorem was proved in [20].

Theorem 2.9. [20] If an edge-coloured graph $G$ has $n$ monochromatic simplices and every vertex of $G$ belongs to exactly one monochromatic simplex of $G$, then $\gamma_{mp}(G) = \ir_{mp}(G) = \Gamma_{mp}(G) = n$.

The monochromatic covering number $\theta_{mp}(G)$ of an edge-coloured graph $G$ is the smallest integer $n$ for which there exists a partition $V_1, \ldots, V_n$ of the vertex set $V(G)$
such that each $V_i$ induces a connected monochromatic subgraph of $G$. It is easy to observe that $\alpha_{mp}(G) \leq \theta_{mp}(G)$.

To prove the next theorem we need the following lemma:

**Lemma 2.10.** Let $G$ be an edge-coloured graph without $p$-coloured cycles, $2 \leq p \leq 4$ and let $S$ and $T$ be disjoint sets of vertices of $G$. If $G[S]$ and $G[T]$ are connected and monochromatic then there is a vertex $s_0 \in S$ such that $N_{G}^{mp}(s_0) \cap T = N_{G}^{mp}(S) \cap T$.

**Proof.** The proof is by induction on $m = |S \cap N_{G}^{mp}(T)|$. If $m \leq 1$ then the result is obvious. Assume that $m > 1$ and that the result is true for all $m' < m$. Let $s \in S \cap N_{G}^{mp}(T)$. By the induction hypothesis there is $s' \in S - \{s\}$ such that $N_{G}^{mp}(s') \cap T = N_{G}^{mp}(S - \{s\}) \cap T$. Evidently if $N_{G}^{mp}(s) \cap T \subseteq N_{G}^{mp}(s') \cap T$ or $N_{G}^{mp}(s') \cap T \subseteq N_{G}^{mp}(s) \cap T$, then $s'$ or $s$, respectively is the desired vertex. To complete the proof it suffices to show that at least one of two sets $N_{G}^{mp}(s) \cap T$ and $N_{G}^{mp}(s') \cap T$ contains the other one. Suppose to the contrary that neither $N_{G}^{mp}(s) \cap T \subseteq N_{G}^{mp}(s') \cap T$ nor $N_{G}^{mp}(s') \cap T \subseteq N_{G}^{mp}(s) \cap T$. Then for every $t \in (N_{G}^{mp}(s) - N_{G}^{mp}(s')) \cap T$ and every $t' \in (N_{G}^{mp}(s') - N_{G}^{mp}(s)) \cap T$ vertices $s, s', t, t'$ belong to a $p$-coloured cycle, $2 \leq p \leq 4$, a contradiction.

This completes the proof of this Lemma. \qed

**Theorem 2.11.** If $G$ is an edge-coloured graph without $p$-coloured cycles, $2 \leq p \leq 4$, then the following statements are equivalent:

1. every vertex of $G$ belongs to exactly one monochromatic simplex
2. $i_{mp}(G) = \alpha_{mp}(G) = \theta_{mp}(G)$.

**Proof.** Let $M_1, ..., M_n$ be the monochromatic simplices of $G$. If every vertex of $G$ belongs to exactly one of them, then by Theorem 2.9 we have that $i_{mp}(G) = \alpha_{mp}(G) = n$ and consequently $\theta_{mp}(G) \leq n$. From this fact and by $\alpha_{mp}(G) \leq \theta_{mp}(G)$ we have that $\alpha_{mp}(G) = \theta_{mp}(G)$. This proves the first implication. To prove the converse implication assume that $M_1, ..., M_n$ are monochromatic subgraphs covering $G$, where $n = \theta_{mp}(G) = \alpha_{mp}(G) = i_{mp}(G)$. Firstly we shall show that $M_1, ..., M_n$ are mutually disjoint.

Suppose on contrary that $v \in M_i \cap M_j$ where ($i \neq j$) and assume that $S$ is any maximal imp-set of $G$ such that $v \in S$. Because $|S \cap (V(M_i) \cup V(M_j))| = 1$ and $|S \cap V(M_k)| \leq 1$ for $k = 1, ..., n$ we have that $|M| \leq n - 1 < \alpha_{mp}(G)$, a contradiction with the maximality of $S$. Next we prove that $M_1, ..., M_n$ are monochromatic simplices of the graph $G$.

Assume on the contrary that at least one of the monochromatic subgraphs is not a monochromatic simplex of $G$. Without loss of generality we can assume that $M_0$ is not a monochromatic simplex of $G$. Clearly $n \geq 2$ and for every vertex $x \in V(M_0)$ there is a monochromatic path $x...y$ to some vertex $y$ of $V(G) - V(M_0)$ and $c(xy) \neq c(x)$. Let $S$ be any minimal subset of $V(G) - V(M_0)$ such that $V(M_0) \subseteq N_{G}^{mp}(S)$, say $|S| = k$. We shall show that the set $S$ is an imp-set in $G$. Suppose on the contrary that $S$ is not an imp-set. Hence there exist $u, v \in S$ and a monochromatic path $u...v$ in $G$. Applying Lemma 2.10 to sets $\{u, ..., v\}$ and $V(M_0)$ we have that there is a vertex $s_0 \in \{u, ..., v\}$ such that $N_{G}^{mp}(\{u, ..., v\}) \cap V(M_0) = N_{G}^{mp}(s_0) \cap V(M_0)$. Clearly $V(M_0) \subseteq N_{G}^{mp}(S - \{u, v\}) \cup s_0$). We consider the following cases:
1. \((S - \{u, v\}) \cup \{s_0\}\) is an imp-set of \(G\).

Then we obtain the contradiction with the minimality of \(S\).

2. \((S - \{u, v\}) \cup \{s_0\}\) is not an imp-set of \(G\).

Then there is \(z \in S \setminus \{u, v\}\) and a monochromatic path \(z \ldots s_0\) in \(G\). Applying Lemma 2.10 to sets \(\{z, \ldots, s_0\}\) and \(V(M_n)\) and proving analogously as above we obtain either case 1 or case 2. Using at most \(k - 1\) steps we obtain the contradiction with the maximality of \(S\).

Thus \(S\) is an imp-set and this gives that \(|S \cap V(M_i)| \leq 1\) for \(i = 1, \ldots, n - 1\).

Consequently \(k = |S| \leq n - 1\) and we can assume that \(|S \cap V(M_i)| = 1\) for \(i = 1, \ldots, k\). Let \(J \subseteq V(G) - N^{mp}_G(S)\) be any (possibly empty) imp-set of \(G\). Because

\[
J \subseteq V(G) - N^{mp}_G[S] \subseteq \bigcup_{j=1}^{n-1} V(M_j)
\]

Thus \(J \cap N^{mp}_G[S] = \emptyset\), then \(S \cup J\) is an imp-set of \(G\) and there is an imp-set \(J\) such that \(S \cup J\) is a maximal imp-set in \(G\) and \(|S \cup J| \leq n - 1 < \alpha_{mp}(G)\), which gives a final contradiction. 

3 Parameters of independence domination and irredundance in edge-colored graphs products

It is often easy to work with graphs whose structure can be characterized in terms of smaller and simpler graphs, so many of the existing results come from the study of products of graphs. The operations on graphs allow us to build several families of graphs and in a large family of considered sets can be characterized in terms of smaller and simpler graphs.

In this paper we study edge-colored \(G\)-join \(\sigma(\alpha, G)\) of graphs which preserves considered parameters with respect to related parameters in the product factors. Let \(G\) be an edge-colored graph on \(V(G) = \{x_1, \ldots, x_n\}\), \(n \geq 2\) and \(\alpha = (G_i)_{i \in \{1, \ldots, n\}}\) be a sequence of vertex disjoint edge-colored graphs on \(V(G_i) = \{y_{i1}, \ldots, y_{ip_i}\}, p_i \geq 1, i = 1, \ldots, n\). Then the \(G\)-join of the graph \(G\) and the sequence \(\alpha\) is the graph \(\sigma(\alpha, G)\) such that

\[
V(\sigma(\alpha, G)) = \bigcup_{i=1}^{n} (\{x_i\} \times \{V(G_i)\}) \quad \text{and} \quad E(\sigma(\alpha, G)) = \{(x_i, y_{ij}^p)(x_q, y_{kl}^q)\} \text{-coloured } \psi;
\]

\((x_i = x_q \text{ and } y_{ij}^p, y_{kl}^q \in E(G_i)\text{-coloured } \psi) \text{ or } (x_i x_q \in E(G)\text{-coloured } \psi)\). By \(G^c\) we mean a copy of \(G\) in \(\sigma(\alpha, G)\). It may be noted that if all graphs from the sequence \(\alpha\) have the same vertex set, then from the \(G\)-join we obtain the generalized lexicographic product of the graph \(G\) and the sequence of graphs \(G_i\), i.e. \(\sigma(\alpha, G) = G[G_1, \ldots, G_n]\). If all graphs from the sequence \(\alpha\) are isomorphic to the same graph \(H\), then we obtain the classical product of graphs, namely the composition \(G[H]\) of the graph \(G\) and \(H\).

Let \(X \subseteq V(G)\) and \(X = \{x_{t_1}, \ldots, x_{t_k}\}\), \(1 \leq k \leq n\). If \(G_i = \begin{cases} 2K_1 & \text{if } i=t_j, j=1, \ldots, k \\ K_1 & \text{otherwise} \end{cases}\), then \(\sigma(\alpha, G)\) is the duplication \(G^X\), see [11, 12, 13].

Independent sets and dominating sets in \(G\)-join of digraphs were studied in [11, 12, 13, 1, 2]. Recently interesting concept of \(H\)-kernels in \(G\)-join of digraphs were studied in [9]. It generalizes imp-sets and dmp-sets in edge-colored graphs.
Theorem 3.1. Let $G$ be an edge-coloured graph on $n$ vertices, $n \geq 2$ and $\alpha$ be a sequence of vertex disjoint edge-coloured graphs $G_i$, $i = 1,...,n$. A subset $S^* \subseteq V(\sigma(\alpha,G))$ is a maximal irmp-set of $\sigma(\alpha,G)$ if and only if $S \subseteq V(G)$ is a maximal imp-set of $G$ such that $S^* = \bigcup_{i \in \mathcal{I}} S_i$, where $\mathcal{I} = \{i; x_i \in S\}$ and $S_i \subseteq V(G_i^*)$ and $S_i$ is an arbitrary nonempty 1-element subset of $V(G_i^*)$, for every $i \in \mathcal{I}$.

Theorem 3.2. Let $G$ be an edge-coloured graph on $n$ vertices, $n \geq 2$ and $\alpha$ be a sequence of vertex disjoint edge-coloured graphs $G_i$, $i = 1,...,n$. A subset $Q^* \subseteq V(\sigma(\alpha,G))$ is a minimal dmp-set of $\sigma(\alpha,G)$ if and only if $Q \subseteq V(G)$ is a minimal dmp-set of $G$ such that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$, where $\mathcal{I} = \{i; x_i \in Q\}$, $Q_i \subseteq V(G_i^*)$ and $Q_i$ is an arbitrary nonempty 1-element subset of $V(G_i^*)$, for every $i \in \mathcal{I}$.

For irmp-sets we prove an analogous theorem.

Theorem 3.3. Let $G$ be an edge-coloured graph on $n$ vertices, $n \geq 2$ and $\alpha$ be a sequence of vertex disjoint edge-coloured graphs $G_i$, $i = 1,...,n$. A subset $X^* \subseteq V(\sigma(\alpha,G))$ is a maximal irmp-set of $\sigma(\alpha,G)$ if and only if $X$ is a maximal irmp-set of $G$ such that $X^* = \bigcup_{i \in \mathcal{I}} X_i$, where $\mathcal{I} = \{i; x_i \in X\}$ and $X_i$ is an arbitrary nonempty subset of $V(G_i^*)$, for every $i \in \mathcal{I}$.

Proof. Let $X^*$ be a maximal irmp-set of $\sigma(\alpha,G)$. Denote $X = \{x_i \in V(G); X^* \cap V(G_i^*) \neq \emptyset\}$. First we shall prove that $X$ is not an irmp-set of $G$. This means that there is a vertex $x_i \in X$ such that $I^\text{imp}_{G_i}(x_i, X) = \emptyset$. Hence by the definition of $\sigma(\alpha,G)$ and the set $X$ we have that $X^* \cap V(G_i^*) \neq \emptyset$ and for every $(x_i, y_i^t)$, $1 \leq t \leq p_i$, holds $I^\text{imp}_{G_i}(x_i, y_i^t), X^*) = \emptyset$. Consequently every $(x_i, y_i^t), 1 \leq t \leq p_i$, is a redundant by monochromatic paths, contradicting the irredundance by monochromatic paths of $X^*$. Now we will prove that $X$ is maximal. Suppose on contrary that $X$ is not a maximal irmp-set of $G$. Then there is $x_i \in (V(G) \setminus X)$ such that $X \cup \{x_i\}$ is an irmp-set of $G$. Hence for every $(x_i, y_m)$, $1 \leq m \leq p_i$ the set $X^* \cup \{(x_i, y_m)\}$ would be a greater irmp-set of $\sigma(\alpha,G)$, a contradicting the maximality of $X^*$. Clearly $X^* = \bigcup_{i \in \mathcal{I}} X_i$, where $\mathcal{I} = \{i; x_i \in X\}$. The definition of $\sigma(\alpha,G)$ implies that for every two vertices from each copy $G_i^*$, $i = 1,...,n$ there is a monochromatic path between them in $\sigma(\alpha,G)$. Let $X_i \subseteq V(G_i^*)$. If $|X_i| \geq 2$, then for an arbitrary subset $Y \subseteq X_i$, where $|Y| \geq 2$ and for each $(x_i, y_i^t), (x_i, y_i^q)$ holds $N_{\sigma(\alpha,G)}[(x_i, y_i^p)] = N_{\sigma(\alpha,G)}[(x_i, y_i^q)]$. Consequently one vertex from copy $G_i^*$ can belong to irmp-set of $\sigma(\alpha,G)$. So $X_i$ is an 1-element set containing arbitrary vertex from $V(G_i^*)$, for every $i \in \mathcal{I}$.

Let $X \subseteq V(G)$ be a maximal irmp-set of $G$ and let $X_i$, where $i \in \mathcal{I}$ and $\mathcal{I} = \{i; x_i \in X\}$ be an 1-element set containing an arbitrary vertex from $V(G_i^*)$. We will prove that $X^* = \bigcup_{i \in \mathcal{I}} X_i$ is a maximal irmp-set of $\sigma(\alpha,G)$. It is obvious from the definition of $\sigma(\alpha,G)$ that $X^*$ is an irmp-set of $\sigma(\alpha,G)$. Assume on the contrary that $X^*$ is not a maximal irmp-set of $\sigma(\alpha,G)$. Then there is $(x_i, y_m) \in V(\sigma(\alpha,G) \setminus X^*)$
such that the set $X^* \cup \{(x_t, y_m^t)\}$ is an irmp-set of $\sigma(\alpha, G)$. Consequently $(x_t, y_m^t)$ is not a redundant by monochromatic paths in $X^*$. The definition of $X^*$ implies that $x_t \notin X$ in other case we get a contradiction with the assumption of $S_t, t \in I$. Moreover the definition of $\sigma(\alpha, G)$ gives that $x_t$ is not redundant by monochromatic paths in $X$. So $X \cup \{x_t\}$ is an irmp-set of $G$, a contradiction with the maximality of $X$.

Thus the Theorem is proved. \[\square\]

From the above theorems immediately follows the following results for parameters of independence, domination and irredundence by monochromatic paths in $\sigma(\alpha, G)$.

**Theorem 3.4.** Let $G, G_1, ..., G_n$ be edge-coloured graphs. Then

1. $\alpha_{mp}(\sigma(\alpha, G)) = \alpha_{mp}(G)$
2. $i_{mp}(\sigma(\alpha, G)) = i_{mp}(G)$
3. $\gamma_{mp}(\sigma(\alpha, G)) = \gamma_{mp}(G)$
4. $\Gamma_{mp}(\sigma(\alpha, G)) = \Gamma_{mp}(G)$
5. $ir_{mp}(\sigma(\alpha, G)) = ir_{mp}(G)$
6. $IR_{mp}(\sigma(\alpha, G)) = IR_{mp}(G)$

**References**


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