J o u r n a l of
Mathematics
and Applications

No 35, pp 97-106 (2012)

On parameters of independence, domination and irredundance in edge-coloured graphs and their products

Andrzej Włoch and *Iwona Włoch*

Submitted by: *Jan Stankiewicz*

Abstract: In this paper we study some parameters of domination, independence and irredundance in some edge-coloured graphs and their products. We present several general properties of independent, dominating and irredundance sets in edge-coloured graphs and we give relationships between the independence, domination and irredundant numbers of an edge-coloured graph. We generalize some classical results concerning independence, domination and irredundance in graphs. Moreover we study G-join of edge-coloured graphs which preserves considered parameters with respect to related parameters in product factors.

AMS Subject Classification: *05C69*
Key Words and Phrases: *edge-coloured graph, monochromatic paths, independence, domination, irredundance*

1 Introduction

Consider a finite connected graph G with a vertex set $V(G)$ and an edge set $E(G)$. A *path* from a vertex x_1 to a vertex x_n, $n \geq 2$, in G is a sequence of distinct vertices $x_1, ..., x_n$ such that $x_i x_{i+1} \in E(G)$, for $i = 1, ..., n-1$; we denote it simply by $x_1...x_n$. If $x_1 = x_n$ the path form a *cycle*. An *edge-m-colouring* of G is a mapping $c : E(G) \to \{1, ..., m\}$. We then say that G is *edge-m-coloured* by c. An m-coloured graph G is *monochromatic* if $c(e) = c(f)$ for any $e, f \in E(G)$. We abuse the notation slightly and call $c(G) = c(e)$, for any $e \in E(G)$. A path (cycle) is m-coloured if its edges are coloured using m-colours. A *path* is called *monochromatic* if its edges are coloured alike. The set of all vertices y for which there is a monochromatic path $y...x$ is called *the chromatic neighborhood of* x and is denoted by $N_G^{mp}(x)$. We write $N_G^{mp}[x]$

instead of $N_G^{mp}(x) \cup \{x\}$. For a subset X of $V(G)$ we write $N_G^{mp}(X)$ and $N_G^{mp}[X]$ instead of $\bigcup_{x \in X} N_G^{mp}(x)$ and $\bigcup_{x \in X} N_G^{mp}[X]$, respectively.

A subset $S \subset V(G)$ is said to be *independent by monochromatic paths* of the edge-coloured graph G if for any two different vertices $x, y \in S$ there is no monochromatic path between them. In addition a subset containing only one vertex and the empty set are independent by monochromatic paths. For convenience we will write an *imp-set* of G instead of an independent by monochromatic paths set of G. For any proper edge-colouring of the graph G an imp-set of G is an independent set in the classical sense. Moreover every imp-set of G is independent. The *lower* and *upper independence by monochromatic paths numbers $i_{mp}(G)$ and $\alpha_{mp}(G)$* of G are respectively the minimum and maximum cardinalities of maximal imp-set of vertices of G.

A subset $Q \subseteq V(G)$ is *dominating by monochromatic paths*, shortly dmp-set of the edge-coloured graph G if for each $x \in V(G) \setminus Q$ there exists a monochromatic path $x...y$, for some $y \in Q$. We will write a *dmp-set* of G instead of dominating by monochromatic paths set of G. For proper edge-colouring of the graph G a dmp-set of G is a dominating set of G in the classical sense. Moreover every dominating set of G is a dmp-set. The *lower* and *upper by monochromatic paths numbers $\gamma_{mp}(G)$ and* $\Gamma_{mp}(G)$ of G are respectively the minimum and maximum cardinalities of minimal dmp-set of vertices of G.
Parameters $\gamma_{mp}(G)$ and $\alpha_{mp}(G)$ will be named as the *domination by monochromatic paths* and *independence by monochromatic paths numbers*, respectively.
Let G be an edge-coloured graph and $X \subset V(G)$. For every $x \in X$, define $I_G^{mp}(x, X) = N_G^{mp}[x] - N_G^{mp}[X - \{x\}]$ the set of *private chromatic neighbours of the vertex relative to the set X*. If $I_G^{mp}(x, X) = \emptyset$, then x is said to be *redundant by monochromatic path in X*. A set X of vertices containing no redundant by monochromatic paths vertex is called *irredundant by monochromatic paths*. The *lower and upper irredudance by monochromatic paths number $ir_{mp}(G)$* and $IR_{mp}(G)$ of a graph G are respectively the minimum and maximum cardinalities of maximal irredundant by monochromatic paths set of vertices of G. The parameter $ir_{mp}(G)$ is the *irredundance by monochromatic paths number* of an edge-coloured graph G. In this paper we will write an *irmp-set* of G instead of an irredundant by monochromatic paths set of G. For the proper edge-colouring of the graph G an irmp-set of G is an irredundant set in the classical sense.
Note that for the proper edge-colouring of the graph G we have the following equalities: $\alpha_{mp}(G) = \alpha(G)$, $\gamma_{mp}(G) = \gamma(G)$, $\Gamma_{mp}(G) = \Gamma(G)$, $i_{mp}(G) = i(G)$, $ir_{mp}(G) = ir(G)$ and $IR_{mp}(G) = IR(G)$.

The concepts of independence, domination and irredundance have existed in literature for a long time, see [14]. There are several generalizations of these concepts, for instance generalization in distance sense see [10, 13].
Concept of independence and domination by monochromatic path in graphs were studied in [1-8] and [15-21]. More generalized concept was considered recently in [9].

In this paper we study parameters of independence, domination and irredundance by monochromatic paths in an edge-coloured graphs and their products. We give some general properties of imp-sets, dmp-sets and irmp-sets in an edge-coloured graph and we give some relationships between studied parameters which generalize results for independent sets, dominating sets and irredundance sets in the classical sense.

2 General properties of imp-sets and dmp-sets in graphs

In this section we give some relations between imp-sets, dmp-sets and irmp-sets in graphs.

Theorem 2.1. *[20] For an arbitrary edge-coloured graph G and a subset $S \subset V(G)$ the following conditions are equivalent:*

1. *S is a maximal imp-set od G.*
2. *S is an imp-set of G and a dmp-set of G.*
3. *S is both a maximal imp-set and a minimal dmp-set of G.*

Theorem 2.2. *Let X be an irmp-set of an edge-coloured graph G. If there exists $x \in X$ such that $x \in I_G^{mp}(x, X)$ then $I_G^{mp}(x, X) \nsubseteq N_G^{mp}[v]$, for any $v \in V(G) - N_G^{mp}[X]$.*

Proof. Assume that there exists $x \in X$ such that $I_G^{mp}(x, X) \subseteq N_G^{mp}[v]$ for some $v \in V(G) - N_G^{mp}[X]$. Then $x \in I_G^{mp}(x, X) \subseteq N_G^{mp}[v]$, that is, $v \in N_G^{mp}[x]$, which contradict the choice of the vertex $v \in V(G) - N_G^{mp}[X]$. □

Theorem 2.3. *Let G be an edge-coloured graph and let Q be a dmp-set in G. Then Q is a minimal dmp-set in G if and only if $I_G^{mp}(x, Q) \neq \emptyset$, for each $x \in Q$.*

Proof. If Q is a minimal dmp-set in G, then for each $x \in Q$ we have that $N_G^{mp}[x] \cup N_G^{mp}[Q - \{x\}] = N_G^{mp}[Q] = V(G)$. Since $N_G^{mp}[Q - \{x\}] \subset V(G)$, so $I_G^{mp}(x, Q) \neq \emptyset$. Assume now that Q is a dmp-set in G and $I_G^{mp}(x, Q) \neq \emptyset$, for each $x \in Q$. Suppose on contrary that Q is not minimal. This means that for some $x \in Q$, $Q - \{x\}$ is a dmp-set in G. Therefore $N_G^{mp}[Q - \{x\}] = V(G)$ and since $N_G^{mp}[x] \subseteq V(G)$, so $I_G^{mp}(x, Q) = \emptyset$, contrary to the hypothesis. □

From the definition of an irmp-set and Theorem 2.3 it follows the following relationships between minimal dmp-sets and maximal irmp-sets:

Corollary 1. *Let Q be a dmp-set of an edge-coloured graph G. Then Q is a minimal dmp-set of G if and only if Q is a maximal irmp-set of G.*

In view of the facts that every maximal imp-set of a graph G is a minimal dmp-set and every minimal dmp-set is a maximal irmp-set it follows the following string of inequalities:

Proposition 2.4. *For any edge-coloured graph G,*

$$ir_{mp}(G) \leq \gamma_{mp}(G) \leq i_{mp}(G) \leq \alpha_{mp}(G) \leq \Gamma_{mp}(G) \leq IR_{mp}(G).$$

Theorem 2.5. *If X is a smallest maximal irmp-set in an edge-coloured graph G and X is an imp-set, then*

$$ir_{mp}(G) = \gamma_{mp}(G) = i_{mp}(G).$$

Proof. By Proposition 2.4 we obtain that $ir_{mp}(G) \leq \gamma_{mp}(G) \leq i_{mp}(G)$, so it suffices to prove that $ir_{mp}(G) = i_{mp}(G)$. Suppose on the contrary that $ir_{mp}(G) \neq i_{mp}(G)$. Then $|X| = ir_{mp}(G) < i_{mp}(G)$ and this implies that X is not a maximal imp-set in G. Consequently $V(G) - N_G^{mp}[X] \neq \emptyset$ and for any $x \in V(G) - N_G^{mp}[X]$ the set $X \cup \{x\}$ is an imp-set of G. Therefore by previous considerations $X \cup \{x\}$ is an irmp-set in G, contrary to the maximality of X. □

Theorem 2.6. *Let $G_1, ..., G_n$ be the connected components of an edge-coloured graph G and X be a maximal irmp-set of G and $X_i = X \cap V(G_i)$. Then $X_i \neq \emptyset$ and X_i is a maximal irmp-set of G_i, for each $i = 1, ..., n$.*

Proof. If $X_i = \emptyset$ for some i, $1 \leq i \leq n$ then we can observe that $X \cup \{y\}$ is an irmp-set of G for any $y \in V(G_i)$ which contradicts the maximality of X. This implies that $X_i \neq \emptyset$ for each $i = 1, ..., n$. Because X is a maximal irmp-set of G hence $I_G^{mp}(x, X) \neq \emptyset$, for each $x \in X$ and since $X_i \subseteq X$, so X_i is also an irmp-set of G_i. Suppose that there exists $1 \leq i \leq n$ such that X_i is not a maximal irmp-set of G_i. Then there exists at least one vertex $y \in V(G_i)$ such that $X_i \cup \{y\}$ is also an irmp-set, and consequently we have that $X \cup \{y\}$ is also an irmp-set of G, a contradiction to the maximality of X. □

Theorem 2.7. *Let G be an edge-coloured graph. If X is a maximal irmp-set of G then for any $u \in V(G) - N_G^{mp}[X]$ there exists some $x \in X$ such that*
(1) $I_G^{mp}(x, X) \subseteq N_G^{mp}(u)$,
(2) for $x_1, x_2 \in I_G^{mp}(x, X)$ such that $x_1 \neq x_2$ either there is a monochromatic path $x_1...x_2$ in G or there exist $y_1, y_2 \in X - \{x\}$ such that there is a monochromatic path from x_1 to each vertex of $I_G^{mp}(y_1, X)$ and there is a monochromatic path from x_2 to each vertex of $I_G^{mp}(y_2, X)$.

Proof. (1). From the assumption about maximality of X we obtain that the set $X \cup \{u\}$ is not an irmp-set in G. Consequently $I_G^{mp}(x, X \cup \{u\}) = \emptyset$ for some $x \in X \cup \{x\}$. Since $u \in V(G) - N_G^{mp}[X]$, hence there is no monochromatic path $u...y$, for every $y \in X$, so $u \in I_G^{mp}(u, X \cup \{u\})$ and therefore $x \neq u$. Because $I_G^{mp}(x, X \cup \{u\}) = N_G^{mp}[x] - N_G^{mp}[X \cup \{u\} - \{x\}] = N_G^{mp}[x] - N_G^{mp}[X - \{x\}] - N_G^{mp}[u] = \emptyset$, then $I_G^{mp}(x, X) = N_G^{mp}[x] - N_G^{mp}[X - \{x\}] \subseteq N_G^{mp}[u]$ and this gives $I_G^{mp}(x, X) \subseteq N_G^{mp}(u)$ as $u \notin I_G^{mp}(x, X)$.
(2). Let x_1, x_2 be two distinct vertices of $I_G^{mp}(x, X)$ such that there is no monochromatic path $x_1...x_2$ in G and suppose on the contrary that for x_1 or x_2, say for x_1 and for all $y_i \in X - \{x\}$ there is $z_i \in I_G^{mp}(y_i, X)$ that there are no monochromatic paths $z_i...x_1$ in G. Then $x_2 \in I_G^{mp}(x, X \cup \{x_1\})$, $u \in I_G^{mp}(x_1, X \cup \{x_1\})$, $z_i \in I_G^{mp}(y_i, X \cup \{x_1\})$ for each $y_i \in X - \{x\}$ and therefore $X \cup \{x_1\}$ is an irmp-set in G, which contradicts the maximality of X. □

Theorem 2.8. *Let X be a smallest maximal irmp-set in G. Let $X_0 \subset X$ be a subset such that $X_0 \cup \{x\}$ is an imp-set in G, for each $x \in X$ and $|X_0| = k < |X|$. Then $\gamma_{mp}(G) \leq 2ir_{mp}(G) - k - 1$*

Proof. Because $|X_0| = k < |X|$ so $X - X_0 \neq \emptyset$. Let $X - X_0 = \{x_1, ..., x_n\}$. Clearly $n \geq 2$. For each $x_i \in X - X_0$ we choose any $x'_i \in I_G^{mp}(x_i, X)$ and we define the set $X' = X \cup \{x'_1, ..., x'_n\}$. From the assumption of X_0 we obtain that $x_i \notin I_G^{mp}(x_i, X)$, so $x'_i \neq x_i$ for $i = 1, ..., n$ and therefore $|X'| \leq 2ir_{mp}(G) - k$. Moreover the assumption of the set X_0 implies that for every $x_p \in X - X_0$ there is $x_q \in X - X_0$ and a monochromatic path $x_p...x_q$ in G. We shall show that X' is a dmp-set in G. Assume on the contrary that X' is not a dmp-set and let $u \in V(G) - N_G^{mp}[X']$. Consequently X is not a dmp-set in G. Thus for every $y \in X$ there is no monochromatic path $u...y$ in G. Then Theorem 2.7 (1) gives that $I_G^{mp}(x, X) \subseteq N_G^{mp}(u)$ for some $x \in X$. If $x \in X_0$, then $x \in I_G^{mp}(x, X)$ and there is a monochromatic path $x...u$, which contradicts the assumption. If $x \in X - X_0$, then $x = x_i$ (for some $i \in \{1, ..., n\}$) and by previous considerations $x_i^{'} \in I_G^{mp}(x_i, X)$. Because $I_G^{mp}(x_i, X) \subseteq N_G(u)$ so there is a monochromatic path $x_i^{'}...u$ in G, a contradiction. Therefore X' is a dmp-set. Since $X \subset X'$, hence Corollary 1 implies that X' is not a minimal dmp-set. Consequently $\gamma_{mp}(G) < |X'| \leq 2ir_{mp}(G) - k$ and $\gamma_{mp} \leq 2ir_{mp}(G) - k - 1$. □

Corollary 2. *For any edge-coloured graph G, $\gamma_{mp}(G) \leq 2ir_{mp}(G) - 1$.*

Proof. Let X be a smallest maximal irmp-set in G. If X is an imp-set then by Proposition 2.4 we have that $\gamma_{mp}(G) \leq ir_{mp}(G)$ and therefore $\gamma_{mp}(G) \leq 2ir_{mp}(G) - 1$. If X is not an imp-set and $G[X]$ has a subset X_0 on k vertices such that $X_0 \cup \{x\}$ is an imp-set in G, for each $x \in X$, then by Theorem 2.8 it follows that $\gamma_{mp}(G) \leq 2ir_{mp}(G) - k - 1 \leq 2ir_{mp}(G) - 1$. □

A vertex x of an edge-coloured graph is called *a monochromatic vertex* if it belongs to exactly one maximal (with respect to set inclusion) connected monochromatic subgraph of G. A connected monochromatic subgraph of a graph G containing at least one monochromatic vertex is called *a monochromatic simplex of* G. Note that if x is a monochromatic vertex of G then $G[N_G^{mp}[x]]$ contains a subgraph being the unique monochromatic simplex of G containing x. A graph G is *monochromatic simplical* if every vertex of G is a monochromatic vertex or belong to the monochromatic simplex. Certainly, if G is a monochromatic simplical graph and $M_1, ..., M_n$ are the monochromatic simplices in G, then $V(G) = \bigcup_{i=1}^{n} V(M_i)$.

The following theorem was proved in [20].

Theorem 2.9. *[20] If an edge-coloured graph G has n monochromatic simplices and every vertex of G belongs to exactly one monochromatic simplex of G, then $\gamma_{mp}(G) = i_{mp}(G) = \alpha_{mp}(G) = \Gamma_{mp}(G) = n$*

The *monochromatic covering number* $\theta_{mp}(G)$ of an edge-coloured graph G is the smallest integer n for which there exists a partition $V_1, ..., V_n$ of the vertex set $V(G)$

such that each V_i induces a connected monochromatic subgraph of G. It is easy to observe that $\alpha_{mp}(G) \leq \theta_{mp}(G)$.

To prove the next theorem we need the following lemma:

Lemma 2.10. *Let G be an edge-coloured graph without p-coloured cycles, $2 \leq p \leq 4$ and let S and T be disjoint sets of vertices of G. If $G[S]$ and $G[T]$ are connected and monochromatic then there is a vertex $s_0 \in S$ such that $N_G^{mp}(s_0) \cap T = N_G^{mp}(S) \cap T$.*

Proof. The proof is by induction on $m = |S \cap N_G^{mp}(T)|$. If $m \leq 1$ then the result is obvious. Assume that $m > 1$ and that the result is true for all $m' < m$. Let $s \in S \cap N_G^{mp}(T)$. By the induction hypothesis there is $s' \in S - \{s\}$ such that $N_G^{mp}(s') \cap T = N_G^{mp}(S - \{s\}) \cap T$. Evidently if $N_G^{mp}(s) \cap T \subseteq N_G^{mp}(s') \cap T$ or $N_G^{mp}(s') \cap T \subseteq N_G^{mp}(s) \cap T$, then s' or s, respectively is the desired vertex. To complete the proof it suffices to show that at least one of two sets $N_G^{mp}(s) \cap T$ and $N_G^{mp}(s') \cap T$ contains the other one. Suppose to the contrary that neither $N_G^{mp}(s) \cap T \subseteq N_G^{mp}(s') \cap T$ nor $N_G^{mp}(s') \cap T \subseteq N_G^{mp}(s) \cap T$. Then for every $t \in (N_G^{mp}(s) - N_G^{mp}(s')) \cap T$ and every $t' \in (N_G^{mp}(s') - N_G^{mp}(s)) \cap T$ vertices s, s', t, t' belong to a p-coloured cycle, $2 \leq p \leq 4$, a contradiction.

This completes the proof of this Lemma. □

Theorem 2.11. *If G is an edge-coloured graph without p-coloured cycles, $2 \leq p \leq 4$, then the following statements are equivalent:*
(1) every vertex of G belongs to exactly one monochromatic simplex
(2) $i_{mp}(G) = \alpha_{mp}(G) = \theta_{mp}(G)$.

Proof. Let $M_1, ..., M_n$ be the monochromatic simplices of G. If every vertex of G belongs to exactly one of them, then by Theorem 2.9 we have that $i_{mp}(G) = \alpha_{mp}(G) = n$ and consequently $\theta_{mp}(G) \leq n$. From this fact and by $\alpha_{mp}(G) \leq \theta_{mp}(G)$ we have that $\alpha_{mp}(G) = \theta_{mp}(G)$. This proves the first implication. To prove the converse implication assume that $M_1, ..., M_n$ are monochromatic subgraphs covering G, where $n = \theta_{mp}(G) = \alpha_{mp}(G) = i_{mp}(G)$. Firstly we shall show that $M_1, ..., M_n$ are mutually disjoint

Suppose on contrary that $v \in M_i \cap M_j$ where $(i \neq j)$ and assume that S is any maximal imp-set of G such that $v \in S$. Because $|S \cap (V(M_i) \cup V(M_j))| = 1$ and $|S \cap V(M_k)| \leq 1$ for $k = 1, ..., n$ we have that $|M| \leq n - 1 < \alpha_{mp}(G)$, a contradiction with the maximality of S. Next we prove that $M_1, ..., M_n$ are monochromatic simplices of the graph G.

Assume on the contrary that at least one of the monochromatic subgraphs is not a monochromatic simplex of G. Without loos of generality we can assume that M_n is not a monochromatic simplex of G. Clearly $n \geq 2$ and for every vertex $x \in V(M_n)$ there is a monochromatic path $x...y$ to some vertex y of $V(G) - V(M_n)$ and $c(xy) \neq c(M_n)$. Let S be any minimal subset of $V(G) - V(M_n)$ such that $V(M_n) \subseteq N_G^{mp}(S)$, say $|S| = k$. We shall show that the set S is an imp-set in G. Suppose on the contrary that S is not an imp-set. Hence there exist $u, v \in S$ and a monochromatic path $u...v$ in G. Applying Lemma 2.10 to sets $\{u, ..., v\}$ and $V(M_n)$ we have that there is a vertex $s_0 \in \{u, ..., v\}$ such that $N_G^{mp}(\{u, ..., v\}) \cap V(M_n) = N_G^{mp}(s_0) \cap V(M_n)$. Clearly $V(M_n) \subseteq N_G^{mp}((S - \{u, v\}) \cup s_0)$. We consider the following casses:

1. $(S-\{u,v\})\cup\{s_0\}$ is an imp-set of G.
   Then we obtain the contradiction with the minimality of S.
2. $(S-\{u,v\})\cup\{s_0\}$ is not an imp-set of G.
   Then there is $z\in S\setminus\{u,v\}$ and a monochromatic path $z...s_0$ in G. Applying Lemma 2.10 to sets $\{z,...,s_0\}$ and $V(M_n)$ and proving analogously as above we obtain either case 1 or case 2. Using at most $k-1$ steps we obtain the contradiction with the maximality of S.

Thus S is an imp-set and this gives that $|S\cap V(M_i)|\leq 1$ for $i=1,...,n-1$. Consequently $k=|S|\leq n-1$ and we can assume that $|S\cap V(M_i)|=1$ for $i=1,...,k$. Let $J\subseteq V(G)-N_G^{mp}(S)$ be any (possibly empty) imp-set of G. Because $J\subseteq V(G)-N_G^{mp}[J]\subseteq\sum\limits_{j=k+1}^{n-1}V(M_j)$ so it immediately follows that $|J|\leq n-k-1$. Moreover, since $J\cap N_G^{mp}[S]=\emptyset$, then $S\cup J$ is an imp-set of G and there is an imp-set J such that $S\cup J$ is a maximal imp-set in G and $|S\cup J|\leq n-1<\alpha_{mp}(G)$, which gives a final contradiction. □

3 Parameters of independence domination and irredundance in edge-coloured graphs products

It is often easy to work with graphs whose structure can be characterized in terms of smaller and simpler graphs, so many of the existing results come from the study of products of graphs. The operations on graphs allow us to build several families of graphs and in a large family of considered sets can be characterized in therms of smaller and simpler graphs.

In this paper we study edge-coloured G-join $\sigma(\alpha,G)$ of graphs which preserves considered parameters with respect to related parameters in the product factors. Let G be and edge-coloured graph on $V(G)=\{x_1,...,x_n\}$, $n\geq 2$ and $\alpha=(G_i)_{i\in\{1,...,n\}}$ be a sequence of vertex disjoint edge-coloured graphs on $V(G_i)=\{y_1,...,y_{p_i}\}$, $p_i\geq 1$, $i=1,...,n$. Then the G-join of the graph G and the sequence α is the graph $\sigma(\alpha,G)$ such that $V(\sigma(\alpha,G))=\bigcup\limits_{i=1}^{n}(\{x_i\}\times\{V(G_i))$ and $E(\sigma(\alpha,G))=\{(x_s,y_j^s)(x_q,y_t^q)$-coloured ψ; $(x_s=x_q$ and $y_j^s y_t^s\in E(G_s)$-coloured $\psi)$ or $(x_sx_q\in E(G)$-coloured $\psi)\}$. By G_i^c we mean a copy of G_i in $\sigma(\alpha,G)$. It may be noted that if all graphs from the sequence α have the same vertex set, then from the G-join we obtain the generalized lexicographic product of the graph G and the sequence of graphs G_i, i.e. $\sigma(\alpha,G)=G[G_1,...,G_n]$. If all graphs from the sequence α are isomorphic to the same graph H, then we obtain the classical product of graphs, namely the composition $G[H]$ of the graph G and H.

Let $X\subseteq V(G)$ and $X=\{x_{t_1},...,x_{t_k}\}$, $1\leq k\leq n$. If $G_i=\begin{cases}2K_1 & \textit{and } i=t_j,\ j=1,...,k\\ K_1 & \textit{otherwise},\end{cases}$

then $\sigma(\alpha,G)$ is the duplication G^X, see [11, 12, 13].

Independent sets and dominating sets in G-join of digraphs were studied in [11, 12, 13, 1, 2]. Recently interesting concept of H-kernels in G-join of digraphs were studied in [9]. It generalize imp-sets and dmp-sets in edge coloured graphs.

Imp-sets and dmp-sets in G-join of digraphs were studied in [9]. Using the same method we can prove similar results for maximal imp-sets and minimal dmp-sets.

Theorem 3.1. *Let G be an edge-coloured graph on n vertices, $n \geq 2$ and α be a sequence of vertex disjoint edge-coloured graphs G_i, $i = 1, ..., n$. A subset $S^* \subset V(\sigma(\alpha, G))$ is a maximal imp-set of $\sigma(\alpha, G)$ if and only if $S \subset V(G)$ is a maximal imp-set of G such that $S^* = \bigcup_{i \in \mathcal{I}} S_i$, where $\mathcal{I} = \{i; x_i \in S\}$ and $S_i \subseteq V(G_i^c)$ and S_i is an arbitrary nonempty 1-element subset of $V(G_i^c)$, for every $i \in \mathcal{I}$.*

Theorem 3.2. *Let G be an edge-coloured graph on n vertices, $n \geq 2$ and α be a sequence of vertex disjoint edge-coloured graphs G_i, $i = 1, ..., n$. A subset $Q^* \subset V(\sigma(\alpha, G))$ is a minimal dmp-set of $\sigma(\alpha, G)$ if and only if $Q \subseteq V(G)$ is a minimal dmp-set of G such that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$, where $\mathcal{I} = \{i; x_i \in Q\}$, $Q_i \subseteq V(G_i^c)$ and Q_i is an arbitrary nonempty 1-element subset of $V(G_i^c)$, for every $i \in \mathcal{I}$.*

For irmp-sets we prove an analogous theorem.

Theorem 3.3. *Let G be an edge-coloured graph on n vertices, $n \geq 2$ and α be a sequence of vertex disjoint edge-coloured graphs G_i, $i = 1, ..., n$. A subset $X^* \subset V(\sigma(\alpha, G))$ is a maximal irmp-set of $\sigma(\alpha, G)$ if and only if X is a maximal irmp-set of G such that $X^* = \bigcup_{i \in \mathcal{I}} X_i$, where $\mathcal{I} = \{i; x_i \in X\}$ and X_i is an arbitrary nonempty subset of $V(G_i^c)$, for every $i \in \mathcal{I}$.*

Proof. Let X^* be a maximal irmp-set of $\sigma(\alpha, G)$. Denote $X = \{x_i \in V(G); X^* \cap V(G_i^c) \neq \emptyset\}$. First we shall prove that X is not an irmp-set of G. This means that there is a vertex $x_i \in X$ such that $I_G^{mp}(x_i, X) = \emptyset$. Hence by the definition of $\sigma(\alpha, G)$ and the set X we have that $X^* \cap V(G_i^c) \neq \emptyset$ and for every (x_i, y_t^i), $1 \leq t \leq p_i$ holds $I_G^{mp}((x_i, y_p^i), X^*) = \emptyset$. Consequently every (x_i, y_t^i), $1 \leq t \leq p_i$, is a redundant by monochromatic paths, contradicting the irredundance by monochromatic paths of X^*. Now we will prove that X is maximal. Suppose on contrary that X is not a maximal irmp-set of G. Then there is $x_t \in (V(G) \setminus X)$ such that $X \cup \{x_t\}$ is an irmp-set of G. Hence for every (x_t, y_m), $1 \leq m \leq p_t$ the set $X^* \cup \{(x_t, y_m)\}$ would be a greater irmp-set of $\sigma(\alpha, G)$, a contradicting the maximality of X^*. Clearly $X^* = \bigcup_{i \in \mathcal{I}} X_i$, where $\mathcal{I} = \{i; x_i \in X\}$. The definition of $\sigma(\alpha, G)$ implies that for every two vertices from each copy G_i^c, $i = 1, ..., n$ there is a monochromatic path between them in $\sigma(\alpha, G)$. Let $X_i \subset V(G_i^c)$. If $|X_i| \geq 2$, then for an arbitrary subset $Y \subseteq X_i$, where $|Y| \geq 2$ and for each $(x_i, y_p^i), (x_i, y_q^i)$ holds $N_{\sigma(\alpha,G)}[(x_i, y_p^i)] = N_{\sigma(\alpha,G)}[(x_i, y_q^i)]$. Consequently one vertex from copy G_i^c can belong to irmp-set of $\sigma(\alpha, G)$. So X_i is an 1-element set containing arbitrary vertex from $V(G_i^c)$, for every $i \in \mathcal{I}$.

Let $X \subseteq V(G)$ be a maximal irmp-set of G and let X_i, where $i \in \mathcal{I}$ and $\mathcal{I} = \{i; x_i \in X\}$ be an 1-element set containing an arbitrary vertex from $V(G_i^c)$. We will prove that $X^* = \bigcup_{i \in \mathcal{I}} X_i$ is a maximal irmp-set of $\sigma(\alpha, G)$. It is obvious from the definition of $\sigma(\alpha, G)$ that X^* is an irmp-set of $\sigma(\alpha, G)$. Assume on the contrary that X^* is not a maximal irmp-set of $\sigma(\alpha, G)$. Then there is $(x_t, y_m^t) \in (V(\sigma(\alpha, G) \setminus X^*)$

such that the set $X^* \cup \{(x_t, y_m^t)\}$ is an irmp-set of $\sigma(\alpha, G)$. Consequently (x_t, y_m^t) is not a redundant by monochromatic paths in X^*. The definition of X^* implies that $x_t \notin X$ in other case we get a contradiction with the assumption of S_t, $t \in \mathcal{I}$. Moreover the definition of $\sigma(\alpha, G)$ gives that x_t is not redundant by monochromatic paths in X. So $X \cup \{x_t\}$ is an irmp-set of G, a contradiction with the maximality of X.

Thus the Theorem is proved. □

From the above theorems immediately follows the following results for parameters of independence, domination ond irredundence by monochromatic paths in $\sigma(\alpha, G)$.

Theorem 3.4. *Let $G, G_1, ..., G_n$ be edge-coloured graphs. Then*

1. $\alpha_{mp}(\sigma(\alpha, G)) = \alpha_{mp}(G)$
2. $i_{mp}(\sigma(\alpha, G)) = i_{mp}(G)$
3. $\gamma_{mp}(\sigma(\alpha, G)) = \gamma_{mp}(G)$
4. $\Gamma_{mp}(\sigma(\alpha, G)) = \Gamma_{mp}(G)$
5. $ir_{mp}(\sigma(\alpha, G)) = ir_{mp}(G)$
6. $IR_{mp}(\sigma(\alpha, G)) = IR_{mp}(G)$

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DOI: 10.7862/rf.2012.9

Andrzej Włoch
email: awloch@prz.edu.pl

Iwona Włoch
email: iwloch@prz.edu.pl

Faculty of Mathematics and Applied Physics
Rzeszów University of Technology
Al. Powstańców Warszawy 12, 35-359 Rzeszów, Poland
Received 17.11.2011