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Fast Growing Solutions to Linear Differential Equations with Entire Coefficients Having the Same ρ_φ-order

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ABSTRACT: This paper deals with the growth of solutions of a class of higher order linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \ k \geq 2$$

when most coefficients $A_j(z)$ $(j = 0, ..., k-1)$ have the same ρ_φ-order with each other. By using the concept of τ_φ-type, we obtain some results which indicate growth estimate of every non-trivial entire solution of the above equations by the growth estimate of the coefficient $A_0(z)$. We improve and generalize some recent results due to Chyzhykov-Semochko and the author.

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1. Introduction and main results

Throughout this paper, the term "meromorphic" will mean meromorphic in the complex plane $\mathbb{C}$. Also, we shall assume that readers are familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f)$ (see, [12, 24]). For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$, see [17, 18].

Definition 1.1 ([17]). Let $p \geq 1$ be an integer. The iterated p-order of a meromorphic function f is defined by

$$\rho_p(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f. If f is entire, then the iterated p-order of f is defined by

$$\rho_p(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r, f)}{\log r} = \limsup_{r \longrightarrow +\infty} \frac{\log_{p+1} M(r, f)}{\log r},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$ is the maximum modulus function.

Definition 1.2 ([17]). The finiteness degree of the order of a meromorphic function f is defined by

$$i(f) := \begin{cases} 0, & \text{for } f \text{ rational}, \\ \min\{j \in \mathbb{N} : \rho_j(f) < \infty\}, & \text{for } f \text{ transcendental for which} \\ & \text{some } j \in \mathbb{N} \text{ with } \rho_j(f) < \infty \text{ exists}, \\ +\infty, & \text{for } f \text{ with } \rho_j(f) = +\infty,\ \forall j \in \mathbb{N}. \end{cases}$$

Definition 1.3 Let f be a meromorphic function. Then the iterated p-type of f, with iterated p-order $0 < \rho_p(f) < \infty$ is defined by

$$\tau_p(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_{p-1} T(r, f)}{r^{\rho_p(f)}} \quad (p \geq 1 \text{ is an integer}).$$

If f is an entire function, then the iterated p-type of f, with iterated p-order $0 < \rho_p(f) < \infty$ is defined by

$$\tau_{M,p}(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p M(r, f)}{r^{\rho_p(f)}} \quad (p \geq 1 \text{ is an integer}).$$

Remark 1.1 Note that for $p = 1$, we can have $\tau_{M,1}(f) \neq \tau_1(f)$. For example if $f(z) = e^z$, then $\tau_{M,1}(f) = 1 \neq \tau_1(f) = \frac{1}{\pi}$. However, by Proposition 2.2.2 in [18], we have $\tau_{M,p}(f) = \tau_p(f)$ for $p \geq 2$.

Consider for $k \geq 2$ the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

where $A_0(z) \not\equiv 0, \ldots, A_{k-1}(z)$ are entire functions. It is well-known that all solutions of equation (1.1) are entire functions and if some of the coefficients of (1.1) are transcendental, then (1.1) has at least one solution with order $\rho(f) = +\infty$. As far as we known, Bernal [7] firstly introduced the idea of iterated order to express the fast growth of solutions of complex linear differential equations. Since then, many authors obtained further results on iterated order of solutions of (1.1), see e.g. [2, 8, 9, 17].

In [17] , Kinnunen have investigated the growth of solutions of equation (1.1) and obtained the following theorem.

Theorem A ([17]). *Let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions such that* $i(A_0) = p$ $(0 < p < \infty)$. *If either* $\max\{i(A_j)\colon j = 1, 2, ..., k-1\} < p$ *or* $\max\{\rho_p(A_j)\colon j = 1, 2, ..., k-1\} < \rho_p(A_0)$, *then every solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $i(f) = p+1$ *and* $\rho_{p+1}(f) = \rho_p(A_0)$.

Note that the result of Theorem A occur when there exists only one dominant coefficient. In the case that there are more than one dominant coefficients, the author [2] obtained the following result.

Theorem B ([2]). *Let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions, and let* $i(A_0) = p$ $(0 < p < \infty)$. *Assume that either*

$$\max\{i(A_j) : j = 1, 2, ..., k-1\} < p$$

or

$$\max\{\rho_p(A_j) : j = 1, 2, ..., k-1\} \leq \rho_p(A_0) = \rho \ \ (0 < \rho < +\infty)$$

and

$$\max\{\tau_{M,p}(A_j) : \rho_p(A_j) = \rho_p(A_0)\} < \tau_{M,p}(A_0) = \tau \ \ (0 < \tau < +\infty).$$

Then every solution $f \not\equiv 0$ *of* (1.1) *satisfies* $i(f) = p+1$ *and* $\rho_{p+1}(f) = \rho_p(A_0) = \rho$.

In [15, 16], Juneja, Kapoor and Bajpai have investigated some properties of entire functions of $[p, q]$-order and obtained some results about their growth. In [20], in order to maintain accordance with general definitions of the entire function f of iterated p-order [17, 18], Liu-Tu-Shi gave a minor modification of the original definition of the $[p, q]$-order given in [15, 16]. With this new concept of $[p, q]$-order, Liu, Tu and Shi [20] have considered equation (1.1) with entire coefficients and obtained different results concerning the growth of their solutions. After that, several authors used this new concept to investigate the growth of solutions in the complex plane and in the unit disc [3, 4, 5, 13, 19, 23, 25] . For the unity of notations, we here introduce the concept of $[p, q]$-order, where p, q are positive integers satisfying $p \geq q \geq 1$ (e.g. see, [19, 20]).

Definition 1.4 ([19, 20]). Let $p \geq q \geq 1$ be integers. If f is a transcendental meromorphic function, then the $[p, q]$-order of f is defined by

$$\rho_{[p,q]}(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r, f)}{\log_q r}.$$

It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq \infty$. If f is rational, then $T(r, f) = O(\log r)$, and so $\rho_{[p,q]}(f) = 0$ for any $p \geq q \geq 1$. By Definition 1.4, we have that $\rho_{[1,1]}(f) =$

$\rho_1(f) = \rho(f)$ usual order, $\rho_{[2,1]}(f) = \rho_2(f)$ hyper-order and $\rho_{[p,1]}(f) = \rho_p(f)$ iterated $p-$order.

Remark 1.2 Both definitions of iterated order and of $[p, q]$-order have the disadvantage that they do not cover arbitrary growth, i.e., there exist entire or meromorphic functions of infinite $[p, q]$-order and p-th iterated order for arbitrary $p \in \mathbb{N}$, i.e., of infinite degree, see Example 1.4 in [10].

Recently, Chyzhykov and Semochko [10] have given general definition of growth for an entire function f in the complex plane, which does not have this disadvantage (see [22]) as follows.

As is $[10]$, let Φ be the class of positive unbounded increasing function on $\lceil 1, +\infty)$ such that $\varphi(e^t)$ is slowly growing, i.e.,

$$\forall c > 0 : \lim_{t \to +\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1.$$

We give some properties of functions from the class Φ.

Proposition 1.1 ([10]). *If $\varphi \in \Phi$, then*

$$\forall m > 0,\ \forall k \geq 0 : \lim_{x \to +\infty} \frac{\varphi^{-1}(\log x^m)}{x^k} = +\infty, \tag{1.2}$$

$$\forall \delta > 0 : \lim_{x \to +\infty} \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} = +\infty. \tag{1.3}$$

Remark 1.3 ([10]). If φ is non-decreasing, then (1.3) is equivalent to the definition of the class Φ.

Definition 1.5 ([10]). Let φ be an increasing unbounded function on $\lceil 1, +\infty)$. Then, the orders of the growth of an entire function f are defined by

$$\tilde{\rho}^0_\varphi(f) = \limsup_{r \longrightarrow +\infty} \frac{\varphi(M(r,f))}{\log r},\ \tilde{\rho}^1_\varphi(f) = \limsup_{r \longrightarrow +\infty} \frac{\varphi(\log M(r,f))}{\log r}.$$

If f is meromorphic, then the orders are defined by

$$\rho^0_\varphi(f) = \limsup_{r \longrightarrow +\infty} \frac{\varphi\left(e^{T(r,f)}\right)}{\log r},\ \rho^1_\varphi(f) = \limsup_{r \longrightarrow +\infty} \frac{\varphi(T(r,f))}{\log r}.$$

Remark 1.4 Now, if we suppose that $\varphi(r) = \log\log r$, then it is clear that $\varphi \in \Phi$. In this case, the above definition of orders coincide with definitions of usual order and hyper-order, i.e., if f is entire, then

$$\tilde{\rho}^0_{\log\log}(f) = \limsup_{r \longrightarrow +\infty} \frac{\log\log M(r,f)}{\log r} = \rho(f),$$

$$\tilde{\rho}^{1}_{\log\log}(f) = \limsup_{r\longrightarrow+\infty} \frac{\log\log\log M(r,f)}{\log r} = \rho_2(f).$$

If f is meromorphic, then

$$\rho^{0}_{\log\log}(f) = \limsup_{r\longrightarrow+\infty} \frac{\log\log\left(e^{T(r,f)}\right)}{\log r} = \limsup_{r\longrightarrow+\infty} \frac{\log T(r,f)}{\log r} = \rho(f),$$

$$\rho^{1}_{\log\log}(f) = \limsup_{r\longrightarrow+\infty} \frac{\varphi(T(r,f))}{\log r} = \limsup_{r\longrightarrow+\infty} \frac{\log\log T(r,f)}{\log r} = \rho_2(f).$$

Proposition 1.2 ([10]). *Let $\varphi \in \Phi$ and f be an entire function. Then*

$$\rho^{j}_{\varphi}(f) = \tilde{\rho}^{j}_{\varphi}(f),\ j = 0,1.$$

Now, by Definition 1.5, we can introduce the concepts of μ_φ lower order.

Definition 1.6 Let φ be an increasing unbounded function on $\lceil 1,+\infty)$. Then, the lower orders of the growth of an entire function f are defined by

$$\tilde{\mu}^{0}_{\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\varphi(M(r,f))}{\log r}, \quad \tilde{\mu}^{1}_{\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\varphi(\log M(r,f))}{\log r}.$$

If f is meromorphic, then the orders are defined by

$$\mu^{0}_{\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\varphi\left(e^{T(r,f)}\right)}{\log r}, \quad \mu^{1}_{\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\varphi(T(r,f))}{\log r}.$$

Proposition 1.3 *Let $\varphi \in \Phi$ and f be an entire function. Then*

$$\mu^{j}_{\varphi}(f) = \widetilde{\mu}^{j}_{\varphi}(f),\ j = 0,1.$$

Proof. By using the same proof of Proposition 3.1 in [10] and replacing lim sup by lim inf, we can easily obtain the Proposition 1.3.

Definition 1.7 Let φ be an increasing unbounded function on $[1,+\infty)$. Then, the types of an entire function f with $0 < \tilde{\rho}^{i}_{\varphi}(f) < +\infty$ $(i = 0,1)$ are defined by

$$\tilde{\tau}^{0}_{M,\varphi}(f) = \limsup_{r\longrightarrow+\infty} \frac{\exp\{\varphi(M(r,f))\}}{r^{\tilde{\rho}^{0}_{\varphi}(f)}}, \quad \tilde{\tau}^{1}_{M,\varphi}(f) = \limsup_{r\longrightarrow+\infty} \frac{\exp\{\varphi(\log M(r,f))\}}{r^{\tilde{\rho}^{1}_{\varphi}(f)}}.$$

If f is meromorphic, then the types of f with $0 < \rho^{i}_{\varphi}(f) < +\infty$ $(i = 0,1)$ are defined by

$$\tau^{0}_{\varphi}(f) = \limsup_{r\longrightarrow+\infty} \frac{\exp\left\{\varphi\left(e^{T(r,f)}\right)\right\}}{r^{\rho^{0}_{\varphi}(f)}}, \quad \tau^{1}_{\varphi}(f) = \limsup_{r\longrightarrow+\infty} \frac{\exp\{\varphi(T(r,f))\}}{r^{\rho^{1}_{\varphi}(f)}}.$$

Definition 1.8 Let φ be an increasing unbounded function on $[1,+\infty)$. Then, the lower types of an entire function f with $0 < \tilde{\mu}^{i}_{\varphi}(f) < +\infty$ $(i = 0,1)$ are defined by

$$\underline{\tilde{\tau}}^{0}_{M,\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\exp\{\varphi(M(r,f))\}}{r^{\tilde{\mu}^{0}_{\varphi}(f)}}, \quad \underline{\tilde{\tau}}^{1}_{M,\varphi}(f) = \liminf_{r\longrightarrow+\infty} \frac{\exp\{\varphi(\log M(r,f))\}}{r^{\tilde{\mu}^{1}_{\varphi}(f)}}.$$

If f is meromorphic, then the lower types of f with $0 < \mu_{\varphi}^{i}(f) < +\infty$ $(i = 0, 1)$ are defined by

$$\underline{\tau}_{\varphi}^{0}(f) = \liminf_{r \longrightarrow +\infty} \frac{\exp\left\{\varphi\left(e^{T(r,f)}\right)\right\}}{r^{\mu_{\varphi}^{0}(f)}}, \quad \underline{\tau}_{\varphi}^{1}(f) = \liminf_{r \longrightarrow +\infty} \frac{\exp\left\{\varphi\left(T(r,f)\right)\right\}}{r^{\mu_{\varphi}^{1}(f)}}.$$

Very recently, Bandura, Skaskiv and Filevych in [1, Theorem 7] proved that for an arbitrary entire transcendental function f of infinite order, there exists a strictly increasing positive unbounded and continuously differentiable function φ on $[1, +\infty)$ such that $\tilde{\rho}_{\varphi}^{0}(f) \in (0, +\infty)$. On the other hand, Chyzhykov and Semochko [10], Semochko [21], Belaïdi [6] used the concepts of ρ_{φ}-orders in order to investigate the growth of solutions of linear differential equations in the complex plane and in the unit disc. Examples of such results are the following two theorems.

Theorem C ([10]). *Let $\varphi \in \Phi$ and let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions satisfying* $\max\{\rho_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\} < \rho_{\varphi}^{0}(A_0)$. *Then, every solution $f \not\equiv 0$ of equation* (1.1) *satisfies* $\rho_{\varphi}^{1}(f) = \rho_{\varphi}^{0}(A_0)$.

Theorem D ([6]). *Let $A_0(z), ..., A_{k-1}(z)$ be entire functions, and let $\varphi \in \Phi$. Assume that* $\max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\} < \tilde{\mu}_{\varphi}^{0}(A_0) \leq \tilde{\rho}_{\varphi}^{0}(A_0) < +\infty$. *Then every solution $f \not\equiv 0$ of* (1.1) *satisfies* $\tilde{\mu}_{\varphi}^{0}(A_0) = \tilde{\mu}_{\varphi}^{1}(f) \leq \tilde{\rho}_{\varphi}^{1}(f) = \tilde{\rho}_{\varphi}^{0}(A_0)$.

The main purpose of this paper is to consider the growth of solutions of equation (1.1) with entire coefficients of finite ρ_{φ}-order in the complex plane by using the concept of τ_{φ}-type. We obtain the following results which extend Theorems A-B-C-D.

Theorem 1.1 *Let $A_0(z), ..., A_{k-1}(z)$ be entire functions, and let $\varphi \in \Phi$. Assume that*

$$\max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\} \leq \tilde{\rho}_{\varphi}^{0}(A_0) = \rho < +\infty \ \ (0 < \rho < +\infty)$$

and

$$\max\{\tilde{\tau}_{M,\varphi}^{0}(A_j) : \tilde{\rho}_{\varphi}^{0}(A_j) = \tilde{\rho}_{\varphi}^{0}(A_0)\} < \tilde{\tau}_{M,\varphi}^{0}(A_0) = \tau \ \ (0 < \tau < +\infty).$$

Then every solution $f \not\equiv 0$ of (1.1) *satisfies* $\tilde{\rho}_{\varphi}^{1}(f) = \tilde{\rho}_{\varphi}^{0}(A_0)$.

By using Proposition 1.2, combining Theorem C and Theorem 1.1, we obtain the following result.

Corollary 1.1 *Let $A_0(z), ..., A_{k-1}(z)$ be entire functions, and let $\varphi \in \Phi$. Assume that either*

$$\max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\} < \tilde{\rho}_{\varphi}^{0}(A_0)$$

or

$$\max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\} \leq \tilde{\rho}_{\varphi}^{0}(A_0) = \rho < +\infty \ \ (0 < \rho < +\infty)$$

and

$$\max\{\tilde{\tau}^{0}_{M,\varphi}(A_j) : \tilde{\rho}^{0}_{\varphi}(A_j) = \tilde{\rho}^{0}_{\varphi}(A_0)\} < \tilde{\tau}^{0}_{M,\varphi}(A_0) = \tau \ \ (0 < \tau < +\infty).$$

Then every solution $f \not\equiv 0$ *of* (1.1) *satisfies* $\tilde{\rho}^{1}_{\varphi}(f) = \tilde{\rho}^{0}_{\varphi}(A_0)$.

Theorem 1.2 *Let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions, and let* $\varphi \in \Phi$. *Assume that*

$$\max\{\tilde{\rho}^{0}_{\varphi}(A_j) : j = 1, \ldots, k-1\} \leq \tilde{\mu}^{0}_{\varphi}(A_0) \leq \tilde{\rho}^{0}_{\varphi}(A_0) = \rho < +\infty \ \ \left(\tilde{\mu}^{0}_{\varphi}(A_0) > 0\right)$$

and

$$\tau_1 = \max\{\tilde{\tau}^{0}_{M,\varphi}(A_j) : \tilde{\rho}^{0}_{\varphi}(A_j) = \tilde{\mu}^{0}_{\varphi}(A_0)\} < \tilde{\underline{\tau}}^{0}_{M,\varphi}(A_0) = \tau \ \ (0 < \tau < +\infty).$$

Then every solution $f \not\equiv 0$ *of* (1.1) *satisfies* $\tilde{\rho}^{1}_{\varphi}(f) = \tilde{\rho}^{0}_{\varphi}(A_0) \geq \tilde{\mu}^{1}_{\varphi}(f) = \tilde{\mu}^{0}_{\varphi}(A_0)$.

By combining Theorem D and Theorem 1.2, we obtain the following result.

Corollary 1.2 *Let* $A_0(z), ..., A_{k-1}(z)$ *be entire functions, and let* $\varphi \in \Phi$. *Assume that either*

$$\max\{\tilde{\rho}^{0}_{\varphi}(A_j) : j = 1, \ldots, k-1\} < \tilde{\mu}^{0}_{\varphi}(A_0) \leq \tilde{\rho}^{0}_{\varphi}(A_0) < +\infty$$

or

$$\max\{\tilde{\rho}^{0}_{\varphi}(A_j) : j = 1, \ldots, k-1\} \leq \tilde{\mu}^{0}_{\varphi}(A_0) \leq \tilde{\rho}^{0}_{\varphi}(A_0) = \rho < +\infty \ \ \left(\tilde{\mu}^{0}_{\varphi}(A_0) > 0\right)$$

and

$$\tau_1 = \max\{\tilde{\tau}^{0}_{M,\varphi}(A_j) : \tilde{\rho}^{0}_{\varphi}(A_j) = \tilde{\mu}^{0}_{\varphi}(A_0)\} < \tilde{\underline{\tau}}^{0}_{M,\varphi}(A_0) = \tau \ \ (0 < \tau < +\infty).$$

Then every solution $f \not\equiv 0$ *of* (1.1) *satisfies* $\tilde{\rho}^{1}_{\varphi}(f) = \tilde{\rho}^{0}_{\varphi}(A_0) \geq \tilde{\mu}^{1}_{\varphi}(f) = \tilde{\mu}^{0}_{\varphi}(A_0)$.

2. Some preliminary lemmas

We recall the following definition. The logarithmic measure of a set $F \subset (1, +\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt$, where $\chi_H(t)$ is the characteristic function of a set H. Our proofs depend mainly upon the following lemmas.

Lemma 2.1 ([11]). *Let* f *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then there exist a set* $E_1 \subset (1, \infty)$ *with finite logarithmic measure and a constant* $B > 0$ *that depends only on* α *and* i, j $(0 \leq i < j \leq k)$, *such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup E_1$, *we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left\{\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right\}^{j-i}.$$

Lemma 2.2 *Let $\varphi \in \Phi$ and f be an entire function with $0 < \tilde{\rho}_{\varphi}^{0}(f) = \rho < +\infty$ and type $0 < \tilde{\tau}_{M,\varphi}^{0}(f) < \infty$. Then for any given $\beta < \tilde{\tau}_{M,\varphi}^{0}(f)$, there exists a set $E_2 \subset [1, +\infty)$ that has infinite logarithmic measure, such that for all $r \in E_2$, we have*

$$\varphi(M(r,f)) > \log(\beta r^{\rho}).$$

Proof. By definitions of $\tilde{\tau}_{M,\varphi}^{0}(f)$ type, there exists an increasing sequence $\{r_n\}$, $r_n \to +\infty$ satisfying $\left(1+\frac{1}{n}\right) r_n < r_{n+1}$ and

$$\lim_{r_n \to +\infty} \frac{\exp\{\varphi(M(r_n,f))\}}{r_n^{\rho}} = \tilde{\tau}_{M,\varphi}^{0}(f).$$

Then, there exists a positive integer n_0 such that for all $n \geq n_0$ and for any given ε with $0 < \varepsilon < \tilde{\tau}_{M,\varphi}^{0}(A_0) - \beta$, we have

$$\exp\{\varphi(M(r_n,f))\} > \left(\tilde{\tau}_{M,\varphi}^{0}(f) - \varepsilon\right) r_n^{\rho}. \tag{2.1}$$

For any given $\beta < \tilde{\tau}_{M,\varphi}^{0}(f)$, there exists a positive integer n_1 such that for all $n \geq n_1$, we have

$$\left(\frac{n}{n+1}\right)^{\rho} > \frac{\beta}{\tilde{\tau}_{M,\varphi}^{0}(f) - \varepsilon}. \tag{2.2}$$

Taking $n \geq n_2 = \max\{n_0, n_1\}$. By (2.1) and (2.2) for any $r \in \left[r_n, \left(1+\frac{1}{n}\right) r_n\right]$, we obtain

$$\exp\{\varphi(M(r,f))\} \geq \exp\{\varphi(M(r_n,f))\} > \left(\tilde{\tau}_{M,\varphi}^{0}(f) - \varepsilon\right) r_n^{\rho}$$

$$\geq \left(\tilde{\tau}_{M,\varphi}^{0}(f) - \varepsilon\right) \left(\frac{n}{n+1} r\right)^{\rho} > \beta r^{\rho}.$$

Set $E_2 = \bigcup_{n=n_2}^{+\infty} \left[r_n, \left(1+\frac{1}{n}\right) r_n\right]$, then there holds

$$lm(E_2) = \sum_{n=n_2}^{+\infty} \int_{r_n}^{\left(1+\frac{1}{n}\right) r_n} \frac{dt}{t} = \sum_{n=n_2}^{+\infty} \log\left(1+\frac{1}{n}\right) = +\infty.$$

Lemma 2.3 ([6]). *Let $\varphi \in \Phi$ and $A_0(z), ..., A_{k-1}(z)$ be entire functions. Then, every solution $f \not\equiv 0$ of (1.1) satisfies*

$$\tilde{\rho}_{\varphi}^{1}(f) \leq \max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 0, 1, \ldots, k-1\}.$$

Lemma 2.4 ([6]). *Let $\varphi \in \Phi$ and f be a meromorphic function with $\mu_{\varphi}^{1}(f) < +\infty$. Then there exists a set $E_3 \subset (1, +\infty)$ with infinite logarithmic measure such that for $r \in E_3 \subset (1, +\infty)$, we have for any given $\varepsilon > 0$*

$$T(r,f) < \varphi^{-1}\left(\left(\mu_{\varphi}^{1}(f) + \varepsilon\right) \log r\right).$$

Lemma 2.5 ([14]). *Let* $f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ *be an entire function,* $\mu(r)$ *be the maximum term, i.e.,*

$$\mu(r)=\max\{|a_n|r^n:n=0,1,2,...\},$$

$\nu(r,f)=\nu_f(r)$ *be the central index of* f*, i.e.,* $\nu(r,f)=\max\{m:\mu(r)=|a_m|r^m\}$. *Then*
(i)

$$\mu(r)=\log|a_0|+\int\limits_0^r\frac{\nu_f(t)}{t}dt$$

here we assume that $|a_0|\neq 0$.
(ii) *For* $r<R$

$$M(r,f)<\mu(r)\left\{\nu_f(R)+\frac{R}{R-r}\right\}.$$

Lemma 2.6 ([14, 18]). *Let* f *be a transcendental entire function. Then there exists a set* $E_4\subset(1,+\infty)$ *with finite logarithmic measure such that for all* z *satisfying* $|z|=r\notin E_4$ *and* $|f(z)|=M(r,f)$*, we have*

$$\frac{f^{(n)}(z)}{f(z)}=\left(\frac{\nu_f(r)}{z}\right)^n(1+o(1)),\quad(n\in\mathbb{N}),$$

where $\nu_f(r)$ *is the central index of* f.

Lemma 2.7 [6]. *Let* $\varphi\in\Phi$ *and* f *be an entire function with* $\tilde{\mu}_{\varphi}^{0}(f)<+\infty$. *Then there exists a set* $E_5\subset(1,+\infty)$ *with infinite logarithmic measure such that for* $r\in E_5\subset(1,+\infty)$, *we have for any given* $\varepsilon>0$

$$M(r,f)<\varphi^{-1}\left(\left(\tilde{\mu}_{\varphi}^{0}(f)+\varepsilon\right)\log r\right).$$

3. Proof of Theorem 1.1

Suppose that $f\ (\not\equiv 0)$ is a solution of equation (1.1). From (1.1), we can write

$$|A_0(z)|\leq\left|\frac{f^{(k)}}{f}\right|+|A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right|+\cdots+|A_1(z)|\left|\frac{f'}{f}\right|.\tag{3.1}$$

If $\max\{\tilde{\rho}_{\varphi}^{0}(A_j):j=1,\ldots,k-1\}<\tilde{\rho}_{\varphi}^{0}(A_0)=\rho$, then by Theorem C, we obtain $\tilde{\rho}_{\varphi}^{1}(f)=\tilde{\rho}_{\varphi}^{0}(A_0)$. Suppose that $\max\{\tilde{\rho}_{\varphi}^{0}(A_j):j=1,2,...,k-1\}=\tilde{\rho}_{\varphi}^{0}(A_0)=\rho\ (0<\rho<+\infty)$ and $\max\{\tilde{\tau}_{M,\varphi}^{0}(A_j):\tilde{\rho}_{\varphi}^{0}(A_j)=\tilde{\rho}_{\varphi}^{0}(A_0)\}<\tilde{\tau}_{M,\varphi}^{0}(A_0)=\tau$ $(0<\tau<+\infty)$. First, we prove that $\rho_1=\tilde{\rho}_{\varphi}^{1}(f)\geq\tilde{\rho}_{\varphi}^{0}(A_0)=\rho$. Suppose the contrary $\rho_1=\tilde{\rho}_{\varphi}^{1}(f)<\tilde{\rho}_{\varphi}^{0}(A_0)=\rho$. Then, there exists a set $I\subseteq\{1,2,...,k-1\}$ such that $\tilde{\rho}_{\varphi}^{0}(A_j)=\tilde{\rho}_{\varphi}^{0}(A_0)=\rho\ (j\in I)$ and $\tilde{\tau}_{M,\varphi}^{0}(A_j)<\tilde{\tau}_{M,\varphi}^{0}(A_0)\ (j\in I)$. Thus, we choose α_1, α_2 satisfying

$$\max\{\tilde{\tau}_{M,\varphi}^{0}(A_j):(j\in I)\}<\alpha_1<\alpha_2<\tilde{\tau}_{M,\varphi}^{0}(A_0)=\tau,$$

for sufficiently large r, we have

$$|A_j(z)| \leq \varphi^{-1}(\log(\alpha_1 r^{\rho})) \quad (j \in J) \tag{3.2}$$

and

$$|A_j(z)| \leq \varphi^{-1}\left(\log r^{\beta_1}\right) \leq \varphi^{-1}(\log(\alpha_1 r^{\rho})) \quad (j \in \{1, ..., k-1\} \setminus J), \tag{3.3}$$

where $0 < \beta_1 < \rho$. By Lemma 2.2, there exists a set $E_2 \subset [1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_2$, we have

$$|A_0(z)| = M(r, A_0) > \varphi^{-1}(\log(\alpha_2 r^{\rho})). \tag{3.4}$$

By Lemma 2.1, there exists a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\,[T(2r, f)]^{k+1} \quad (j = 1, 2, ..., k).$$

Since $\tilde{\rho}_{\varphi}^{1}(f) = \rho_1$, then by Proposition 1.2, for any given ε with $0 < \varepsilon < \rho - \rho_1$ and sufficiently large $|z| = r \notin E_1 \cup [0, 1]$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\,[T(2r, f)]^{k+1} \leq B\left[\varphi^{-1}\left(\log(2r)^{\rho_1+\varepsilon}\right)\right]^{k+1} \quad (j = 1, 2, ..., k). \tag{3.5}$$

Hence, by substituting (3.2), (3.3), (3.4) and (3.5) into (3.1), for any given ε $\left(0 < \varepsilon < \min\left\{\frac{\alpha_2 - \alpha_1}{2}, \rho - \rho_1\right\}\right)$ and for sufficiently large $|z| = r \in E_2 \setminus (E_1 \cup [0, 1])$, we have

$$\begin{aligned}\varphi^{-1}(\log(\alpha_2 r^{\rho})) &\leq kB\varphi^{-1}(\log(\alpha_1 r^{\rho}))\left[\varphi^{-1}\left(\log(2r)^{\rho_1+\varepsilon}\right)\right]^{k+1} \\ &\leq \varphi^{-1}(\log((\alpha_1 + 2\varepsilon) r^{\rho})).\end{aligned} \tag{3.6}$$

The latter two estimates follow from the properties of (1.2) and (1.3). Since $E_2 \setminus (E_1 \cup [0, 1])$ is a set of infinite logarithmic measure, then there exists a sequence of points $|z_n| = r_n \in E_2 \setminus (E_1 \cup [0, 1])$ tending to $+\infty$. It follows by (3.6) that

$$\varphi^{-1}(\log(\alpha_2 r_n^{\rho})) \leq \varphi^{-1}(\log((\alpha_1 + 2\varepsilon) r_n^{\rho}))$$

holds for all z_n satisfying $|z_n| = r_n \in E_2 \setminus (E_1 \cup [0, 1])$ as $|z_n| \to +\infty$. By arbitrariness of $\varepsilon > 0$ and the monotonicity of the function φ^{-1}, we obtain that $\alpha_1 \geq \alpha_2$. This contradiction proves the inequality $\tilde{\rho}_{\varphi}^{1}(f) \geq \tilde{\rho}_{\varphi}^{0}(A_0)$. On the other hand, by Lemma 2.3, we have

$$\tilde{\rho}_{\varphi}^{1}(f) \leq \max\{\tilde{\rho}_{\varphi}^{0}(A_j) : j = 0, 1, \ldots, k-1\} = \tilde{\rho}_{\varphi}^{0}(A_0).$$

Hence, every solution $f \not\equiv 0$ of equation (1.1) satisfies $\tilde{\rho}_{\varphi}^{1}(f) = \tilde{\rho}_{\varphi}^{0}(A_0)$.

4. Proof of Theorem 1.2

Suppose that f ($\not\equiv 0$) is a solution of equation (1.1). Then by Theorem 1.1, we obtain $\tilde{\rho}_{\varphi}^{1}(f) = \tilde{\rho}_{\varphi}^{0}(A_0)$. Now, we prove that $\mu_1 = \tilde{\mu}_{\varphi}^{1}(f) \geq \tilde{\mu}_{\varphi}^{0}(A_0) = \mu$. Suppose the contrary $\mu_1 = \tilde{\mu}_{\varphi}^{1}(f) < \tilde{\mu}_{\varphi}^{0}(A_0) = \mu$. We set $b = \max\{\tilde{\rho}_{\varphi}^{0}(A_j) : \tilde{\rho}_{\varphi}^{0}(A_j) < \tilde{\mu}_{\varphi}^{0}(A_0)\}$. If $\tilde{\rho}_{\varphi}^{0}(A_j) < \tilde{\mu}_{\varphi}^{0}(A_0)$, then for any given ε with $0 < 3\varepsilon < \min\{\mu - b, \tau - \tau_1\}$ and for sufficiently large r, we have

$$|A_j(z)| \leq \varphi^{-1}\left(\log r^{b+\varepsilon}\right) \leq \varphi^{-1}\left(\log r^{\tilde{\mu}_{\varphi}^{0}(A_0)-2\varepsilon}\right). \tag{4.1}$$

If $\tilde{\rho}_{\varphi}^{0}(A_j) = \tilde{\mu}_{\varphi}^{0}(A_0)$, $\tilde{\tau}_{M,\varphi}^{0}(A_j) \leq \tau_1 < \underline{\tilde{\tau}}_{M,\varphi}^{0}(A_0) = \tau$, then for sufficiently large r, we have

$$|A_j(z)| \leq \varphi^{-1}\left(\log(\tau_1+\varepsilon) r^{\tilde{\mu}_{\varphi}^{0}(A_0)}\right) \tag{4.2}$$

and

$$|A_0(z)| \geq \varphi^{-1}\left(\log(\tau-\varepsilon) r^{\tilde{\mu}_{\varphi}^{0}(A_0)}\right). \tag{4.3}$$

From (1.1), we can write

$$|A_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right|. \tag{4.4}$$

By Lemma 2.1, there exists a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1 \cup [0,1]$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(2r,f)\right]^{k+1} \quad (j = 1, 2, ..., k).$$

By Proposition 1.3 and Lemma 2.4, for any given ε with $0 < \varepsilon < \mu - \mu_1$ and sufficiently large $|z| = r \in E_3 \backslash (E_1 \cup [0,1])$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(2r,f)\right]^{k+1} < B\left[\varphi^{-1}\left(\log(2r)^{\mu_1+\varepsilon}\right)\right]^{k+1} \quad (j = 1, 2, ..., k), \tag{4.5}$$

where E_3 is a set of infinite logarithmic measure. Hence, by substituting (4.1) – (4.3) and (4.5) into (4.4), for the above ε with $0 < \varepsilon < \min\left\{\frac{\mu-b}{3}, \frac{\tau-\tau_1}{3}, \mu-\mu_1\right\}$), we obtain for sufficiently large $|z| = r \in E_3 \backslash (E_1 \cup [0,1])$

$$\begin{aligned}\varphi^{-1}\left(\log(\tau-\varepsilon) r^{\tilde{\mu}_{\varphi}^{0}(A_0)}\right) &\leq Bk\varphi^{-1}\left(\log(\tau_1+\varepsilon) r^{\tilde{\mu}_{\varphi}^{0}(A_0)}\right)\left[T(2r,f)\right]^{k+1} \\ &\leq Bk\varphi^{-1}\left(\log(\tau_1+\varepsilon) r^{\tilde{\mu}_{\varphi}^{0}(A_0)}\right)\left[\varphi^{-1}\left(\log(2r)^{\mu_1+\varepsilon}\right)\right]^{k+1} \\ &\leq \varphi^{-1}\left(\log(\tau_1+2\varepsilon) r_n^{\tilde{\mu}_{\varphi}^{0}(A_0)}\right). \end{aligned} \tag{4.6}$$

The latter two estimates follow from the properties of (1.2) and (1.3). Since $E_3\backslash(E_1\cup[0,1])$ is a set of infinite logarithmic measure, then there exists a sequence of points $|z_n|=r_n\in E_3\backslash(E_1\cup[0,1])$ tending to $+\infty$. It follows by (4.6) that

$$\varphi^{-1}\left(\log(\tau-\varepsilon)\,r_n^{\tilde{\mu}_\varphi^0(A_0)}\right)\le\varphi^{-1}\left(\log(\tau_1+2\varepsilon)\,r_n^{\tilde{\mu}_\varphi^0(A_0)}\right)$$

holds for all z_n satisfying $|z_n|=r_n\in E_3\backslash(E_1\cup[0,1])$ as $|z_n|\to+\infty$. By arbitrariness of $\varepsilon>0$ and the monotonicity of the function φ^{-1}, we obtain that $\tau_1\ge\tau$. This contradiction proves the inequality $\tilde{\mu}_\varphi^1(f)\ge\tilde{\mu}_\varphi^0(A_0)$.

Now, we prove $\tilde{\mu}_\varphi^1(f)\le\tilde{\mu}_\varphi^0(A_0)$. By (1.1), we have

$$\left|\frac{f^{(k)}}{f}\right|\le|A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right|+\cdots+|A_1(z)|\left|\frac{f'}{f}\right|+|A_0(z)|. \tag{4.7}$$

By Lemma 2.6, there exists a set $E_4\subset(1,+\infty)$ of finite logarithmic measure such that the estimation

$$\frac{f^{(j)}(z)}{f(z)}=\left(\frac{\nu_f(r)}{z}\right)^j(1+o(1))\quad(j=1,...,k) \tag{4.8}$$

holds for all z satisfying $|z|=r\notin E_4$, $r\to+\infty$ and $|f(z)|=M(r,f)$. By Lemma 2.7, for any given $\varepsilon>0$, there exists a set $E_5\subset(1,+\infty)$ that has infinite logarithmic measure, such that for $|z|=r\in E_5$

$$|A_0(z)|<\varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+\varepsilon}\right). \tag{4.9}$$

Substituting (4.1), (4.2), (4.8) and (4.9) into (4.7), we obtain

$$\begin{aligned}\nu_f(r)&\le kr^k|1+o(1)|\,\varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+\varepsilon}\right)\\&\le\varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+2\varepsilon}\right)\end{aligned} \tag{4.10}$$

for all z satisfying $|z|=r\in E_5\backslash E_4$, $r\to+\infty$ and $|f(z)|=M(r,f)$. By (4.10), Lemma 2.5 and Proposition 1.1, we obtain for each $\varepsilon>0$

$$\begin{aligned}T(r,f)&\le\log M(r,f)<\log\mu(r,f)+\log(\nu(2r,f)+2)\\&<2\nu(r,f)\log r+\log(2\nu(2r,f))\\&\le2\varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+2\varepsilon}\right)\log r+\log\left(2\varphi^{-1}\left(\log(2r)^{\tilde{\mu}_\varphi^0(A_0)+\varepsilon}\right)\right)\\&=2\varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+2\varepsilon}\right)\log r+\log2+\log\varphi^{-1}\left(\log(2r)^{\tilde{\mu}_\varphi^0(A_0)+\varepsilon}\right)\\&\le\varphi^{-1}\left(\log r^{\tilde{\mu}_\varphi^0(A_0)+3\varepsilon}\right).\end{aligned}$$

Hence,

$$\frac{\varphi\left(T(r,f)\right)}{\log r} \leq \frac{\log r^{\tilde{\mu}_{\varphi}^{0}(A_0)+3\varepsilon}}{\log r} = \tilde{\mu}_{\varphi}^{0}\left(A_0\right) + 3\varepsilon.$$

It follows

$$\mu_{\varphi}^{1}\left(f\right) = \tilde{\mu}_{\varphi}^{1}\left(f\right) = \liminf_{r\longrightarrow+\infty} \frac{\varphi\left(T(r,f)\right)}{\log r} \leq \tilde{\mu}_{\varphi}^{0}\left(A_0\right) + 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\tilde{\mu}_{\varphi}^{1}\left(f\right) \leq \tilde{\mu}_{\varphi}^{0}\left(A_0\right)$. Hence, every solution $f \not\equiv 0$ of equation (1.1) satisfies $\tilde{\mu}_{\varphi}^{0}\left(A_0\right) = \tilde{\mu}_{\varphi}^{1}\left(f\right) \leq \tilde{\rho}_{\varphi}^{1}\left(f\right) = \tilde{\rho}_{\varphi}^{0}\left(A_0\right)$.

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