Complete convergence under special hypotheses

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Abstract: We prove Baum-Katz type theorems along subsequences of random variables under Komlós-Saks and Mazur-Orlicz type boundedness hypotheses

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1 Introduction and main results

Throughout the paper we shall work with real valued random variables on a complete probability space \((\Omega, F, P)\). The following Baum-Katz type result (cf. [5]) quantifies the rate of convergence in the strong law of large numbers for general sequences of random variables in the form of a complete convergent series:

**Theorem 0.** If \((X_n)_{n \geq 1}\) is an \(L^p\)-norm bounded sequence for some \(0 < p < 2\), i.e., \(\sup_{n \geq 1} \|X_n\|_p \leq C\) for some \(C > 0\), then there exists a subsequence \((Y_n)_{n \geq 1}\) of \((X_n)_{n \geq 1}\) such that, for all \(0 < r \leq p\), we have

\[
\sum_{n=1}^{\infty} n^{p/r - 2} P \left( \left\{ \omega \in \Omega : \left| \sum_{j=1}^{n} Y_j(\omega) \right| > \varepsilon n^{1/r} \right\} \right) < \infty \text{ for } \varepsilon > 0.
\]

In particular the strong law of large numbers holds along the subsequence \((Y_n)_{n \geq 1}\), i.e., \(Y_n/n^{1/p} \to 0\) a.s.

The examples in [6], [4] and [3] show that (1) may fail if one drops the \(L^p\)-norm boundedness hypothesis. Inspired by the celebrated Komlós-Saks and Mazur-Orlicz extensions of the law of large numbers, in this note we shall prove two versions of
the Baum-Katz theorem under special boundedness hypotheses, more general than $L^p$-norm boundedness condition required in Theorem 0.

**Theorem 1.** Let $0 < p < 2$ and $(X_n)_{n \geq 1}$ a sequence such that \( \lim \sup_n |X_n(\omega)|^p < \infty \) for all $\omega \in \Omega$. Then there exists a subsequence $(Y_n)_{n \geq 1}$ of $(X_n)_{n \geq 1}$ such that (1) holds for all $0 < r \leq p$.

**Theorem 2.** Let $0 < p < 2$ and $(X_n)_{n \geq 1}$ a sequence satisfying the following condition: for every subsequence $(\tilde{X}_n)_{n \geq 1}$ of $(X_n)_{n \geq 1}$ and $n \geq 1$, there exists a convex combination $Z_n$ of \{ $|\tilde{X}_n|^p, |\tilde{X}_{n+1}|^p, \ldots$ \}, such that $\limsup_n |Z_n(\omega)| < \infty$ for all $\omega \in \Omega$. Then there exists a subsequence $(Y_n)_{n \geq 1}$ of $(X_n)_{n \geq 1}$ such that (1) holds for all $0 < r \leq p$.

**Remarks.** (i) Both Theorems 1 and 2 hold for uniformly bounded sequences $(X_n)_{n \geq 1}$ in $L^p, 0 < p < 2$. On $[0, 1]$ endowed with the Lebesgue measure, the sequence $X_n(\omega) = n^2$ if $0 \leq \omega \leq 1/n$ and 0 otherwise, satisfies Theorem 2 because $X_n \to 0$ Lebesgue-a.s., yet it does not satisfy Theorem 1 with $p = 1$ because it is not bounded in $L^1[0, 1]$.

(ii) As a matter of fact, both Theorems 1 and 2 may fail for unbounded sequences, e.g., $X_n = n$.

(iii) The idea beneath Theorems 1 and 2 is to construct a rich family of uniformly integrable subsequences of $(X_n)_{n \geq 1}$ as in [2], for which condition (1) holds; note that the hypotheses in [6] and [3] cannot produce Baum-Katz type theorems, as the families of subsequences therein are no longer uniformly integrable.

## 2 Proofs of the Results

**Proof of Theorem 1.** Note that $\lim \sup_n |X_n(\omega)|^p < \infty$ is equivalent to

$$\sup_{n \geq 1} |X_n(\omega)|^p < \infty$$

for all $\omega \in \Omega$. For any natural number $m \geq 1$, let us define

$$A_m = \left\{ \omega \in \Omega : \sup_{n \geq 1} |X_n(\omega)|^p \leq m \right\}.$$

Assume that $r < p$ and fix $a > p/r - 1$. As $P(A_m) \to 1$ as $m \to \infty$, we can choose $m_1 \geq 1$ such that $P(A_{m_1}) > 1 - 2^{-a}$. Integrating and applying Fatou’s lemma, we obtain

$$\sup_{n \geq 1} \int_{A_{m_1}} |X_n(\omega)|^p dP(\omega) \leq m_1. \tag{2}$$

We now apply the Biting Lemma (cf. [1]) to the sequence $(X_n)_{n \geq 1}$ and obtain: an increasing sequence of sets $(B_k^1)_{k \geq 1}$ in $\mathcal{F}$ with $P(B_k^1) \to 1$ as $k \to \infty$, and a subsequence $(X^1_n)_{n \geq 1}$ of $(X_n)_{n \geq 1}$ such that $(X^1_n)_{n \geq 1}$ is uniformly integrable on each
set \( A_{m_1} \cap B_k^1, k \geq 1 \). The latter fact together with estimate (2) show that Theorem 0 applies to the sequence \((X_n^1)_{n \geq 1}\) and gives
\[
\sum_{n=1}^{\infty} n^{p/r-2} P \left( \left\{ \omega \in A_{m_1} \cap B_k^1 : \left| \sum_{j=1}^{n} X_j^1(\omega) \right| > \varepsilon n^{1/r} \right\} \right) < \infty \text{ for } \varepsilon > 0 \text{ and } k \geq 1.
\]

Another application of the Biting Lemma to \((X_n^1)_{n \geq 1}\), instead of \((X_n)_{n \geq 1}\), produces: a measurable set \( A_{m_2} \) with \( P(A_{m_2}) > 1 - 3^{-a} \), an increasing sequence of sets \((B_k^2)_{k \geq 1}\) in \( \mathcal{F} \) with \( P(B_k^2) \to 1 \) as \( k \to \infty \), and a subsequence \((X_n^2)_{n \geq 1}\) of \((X_n^1)_{n \geq 1}\) such that \((X_n^2)_{n \geq 1}\) is uniformly integrable on each set \( A_{m_2} \cap B_k^2, k \geq 1 \), such that
\[
\sum_{n=1}^{\infty} n^{p/r-2} P \left( \left\{ \omega \in A_{m_2} \cap B_k^2 : \left| \sum_{j=1}^{n} X_j^2(\omega) \right| > \varepsilon n^{1/r} \right\} \right) < \infty \text{ for } \varepsilon > 0 \text{ and } k \geq 1.
\]

By induction, we construct for each \( i \geq 1 \): a measurable set \( A_{m_i} \) with \( P(A_{m_i}) > 1 - (i + 1)^{-a} \), an increasing sequence of sets \((B_k^i)_{k \geq 1}\) in \( \mathcal{F} \) with \( P(B_k^i) \to 1 \) as \( k \to \infty \), and a subsequence \((X_n^i)_{n \geq 1}\) of \((X_n^{i-1})_{n \geq 1}\), with the convention that \((X_n^0)_{n \geq 1}\) is precisely \((X_n)_{n \geq 1}\), such that \((X_n^i)_{n \geq 1}\) is uniformly integrable on each set \( A_{m_i} \cap B_k^i, k \geq 1 \), and
\[
\sum_{n=1}^{\infty} n^{p/r-2} P \left( \left\{ \omega \in A_{m_i} \cap B_k^i : \sum_{j=1}^{n} X_j^i(\omega) > \varepsilon n^{1/r} \right\} \right) < \infty \text{ for } \varepsilon > 0 \text{ and } k, i \geq 1.
\]

Now define \( Y_n := X_n^0 \) and, using a diagonal argument in the above formula, we obtain that
\[
\sum_{n=1}^{\infty} n^{p/r-2} P \left( \left\{ \omega \in A_{m_i} \cap B_k^i : \sum_{j=1}^{n} Y_j(\omega) > \varepsilon n^{1/r} \right\} \right) < \infty \text{ for } \varepsilon > 0 \text{ and } k \geq 1. \tag{3}
\]

As \( P(B_k^0) \to 1 \) as \( k \to \infty \) for all \( n \geq 1 \), formula (3) and the dominated convergence theorem imply that
\[
\sum_{n=1}^{\infty} n^{p/r-2} P \left( \left\{ \omega \in A_{m_n} : \sum_{j=1}^{n} Y_j(\omega) > \varepsilon n^{1/r} \right\} \right) < \infty \text{ for } \varepsilon > 0. \tag{4}
\]

Therefore, to prove that series (1) converges for our subsequence \((Y_n)_{n \geq 1}\) and \( r < p \), it suffices to prove (4) with \( A_{m_n} \) replaced by its complement, i.e.,
\[
\sum_{n=1}^{\infty} n^{p/r-2} P \left( \left\{ \omega \in \Omega \setminus A_{m_n} : \sum_{j=1}^{n} Y_j(\omega) > \varepsilon n^{1/r} \right\} \right) < \infty \text{ for } \varepsilon > 0. \tag{5}
\]

Indeed, the latter series is
\[
\leq \sum_{n=1}^{\infty} n^{p/r-2} P \left( \left\{ \omega \in \Omega \setminus A_{m_n} \right\} \right) \leq \sum_{n=1}^{\infty} n^{p/r-2-1} < \infty \tag{6}
\]
as $P(A_m) > 1 - (n + 1)^{-a} > 1 - n^{-a}$ and $a > p/r - 1$. The proof is achieved in the case $r < p$.

If $r = p$, then we modify the induction process as follows: choose measurable sets $A_m$, with $P(A_m) > i/(i + 1)$ for all $i \geq 1$; as such, the diagonal argument above gives the following replacement of (4):

$$\sum_{n=1}^{\infty} \frac{1}{n} \left\{ \omega \in A_{m_n} : \left| \sum_{j=1}^{n} Y_j(\omega) \right| \geq \varepsilon n^{1/r} \right\} < \infty \text{ for } \varepsilon > 0.$$  

To show that series (1) converges for our subsequence $(Y_n)_{n \geq 1}$ and $r = p$, it suffices to prove the following replacement of (5):

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left( \omega \in \Omega \setminus A_{m_n} : \left| \sum_{j=1}^{n} Y_j(\omega) \right| \geq \varepsilon n^{1/r} \right) < \infty \text{ for } \varepsilon > 0.$$  

Indeed, the latter series is

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} P\left( \omega \in \Omega \setminus A_{m_n} \right) \leq \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \infty$$

by the choice of $P(A_{m_n}), n \geq 1$. The latter is the substitute of (6) in the case $r = p$, and the proof is now complete.

**Proof of Theorem 2.** By hypothesis we can write

$$Z_n = \sum_{i \in I_n} \lambda_i^n |\tilde{X}_{n+i}|^p$$

for some $\lambda_i^n \geq 0$ with $\sum_{i \in I_n} \lambda_i^n = 1$,

and where $I_n$ are finite subsets of $\{0, 1, 2, \ldots\}$. In addition, the sequence $(Z_n)_{n \geq 1}$ satisfies the condition $\sup_{n \geq 1} |Z_n(\omega)|^p < \infty$ for all $\omega \in \Omega$. For any natural number $m \geq 1$, let us define $A_m = \left\{ \omega \in \Omega : \sup_{n \geq 1} |Z_n(\omega)| \leq m \right\}$. As $P(A_m) \to 1$ as $m \to \infty$, we can choose $m_1 \geq 1$ such that $P(A_{m_1}) > 1 - 2^{-a}$ or $1/2$, according to $p > r$ or $p = r$, and where $a > p/r - 1$ is fixed. Integrating and applying Fatou’s lemma, we obtain

$$\sup_{n \geq 1} \sum_{i \in I_n} \lambda_i^n \int_{A_{m_1}} |\tilde{X}_{n+i}(\omega)|^p dP(\omega) \leq m_1.$$  

Hence there is a subsequence $(\tilde{X}_n)_{n \geq 1}$ of $(\tilde{X}_n)_{n \geq 1}$ (therefore of $(X_n)_{n \geq 1}$ as well), such that

$$\sup_{n \geq 1} \int_{A_{m_1}} |\tilde{X}_n(\omega)|^p dP(\omega) \leq m_1,$$

which is precisely eq. (2) along a subsequence. The remainder of the proof goes exactly as in the proof of Theorem 1.
References


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