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Complete convergence under special hypotheses

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ABSTRACT: We prove Baum-Katz type theorems along subsequences of random variables under Komlós-Saks and Mazur-Orlicz type boundedness hypotheses

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1 Introduction and main results

Throughout the paper we shall work with real valued random variables on a complete probability space $(\Omega, \mathcal{F}, P)$. The following Baum-Katz type result (cf. [5]) quantifies the rate of convergence in the strong law of large numbers for general sequences of random variables in the form of a complete convergent series:

Theorem 0. *If $(X_n)_{n\geq 1}$ is an L^p-norm bounded sequence for some $0 < p < 2$, i.e., $\sup_{n\geq 1} ||X_n||_p \leq C$ for some $C > 0$, then there exists a subsequence $(Y_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ such that, for all $0 < r \leq p$, we have*

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in \Omega : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0. \tag{1}$$

In particular the strong law of large numbers holds along the subsequence $(Y_n)_{n\geq 1}$, i.e., $Y_n/n^{1/p} \to 0$ a.s.

The examples in [6], [4] and [3] show that (1) may fail if one drops the L^p-norm boundedness hypothesis. Inspired by the celebrated Komlós-Saks and Mazur-Orlicz extensions of the law of large numbers, in this note we shall prove two versions of the Baum-Katz theorem under special boundedness hypotheses, more general than L^p−norm boundedness condition required in Theorem 0.

Theorem 1. *Let $0 < p < 2$ and $(X_n)_{n\geq 1}$ a sequence such that $\limsup_n |X_n(\omega)|^p < \infty$ for all $\omega \in \Omega$. Then there exists a subsequence $(Y_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ such that (1) holds for all $0 < r \leq p$.*

Theorem 2. *Let $0 < p < 2$ and $(X_n)_{n\geq 1}$ a sequence satisfying the following condition: for every subsequence $(\tilde{X}_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ and $n \geq 1$, there exists a convex combination Z_n of $\left\{|\tilde{X}_n|^p, |\tilde{X}_{n+1}|^p, \dots\right\}$, such that $\limsup_n |Z_n(\omega)| < \infty$ for all $\omega \in \Omega$. Then there exists a subsequence $(Y_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ such that (1) holds for all $0 < r \leq p$.*

Remarks. (i) Both Theorems 1 and 2 hold for uniformly bounded sequences $(X_n)_{n\geq 1}$ in $L^p, 0 < p < 2$. On $[0, 1]$ endowed with the Lebesgue measure, the sequence $X_n(\omega) = n^2$ if $0 \leq \omega \leq 1/n$ and 0 otherwise, satisfies Theorem 2 because $X_n \to 0$ Lebesgue-a.s., yet it does not satisfy Theorem 1 with $p = 1$ because it is not bounded in $L^1[0, 1]$. As a matter of fact, both Theorems 1 and 2 may fail for unbounded sequences, e.g., $X_n = n$.

(ii) The idea beneath Theorems 1 and 2 is to construct a rich family of uniformly integrable subsequences of $(X_n)_{n\geq 1}$ as in [2], for which condition (1) holds; note that the hypotheses in [6] and [3] cannot produce Baum-Katz type theorems, as the families of subsequences therein are no longer uniformly integrable.

2 Proofs of the results

Proof of Theorem 1. Note that $\limsup_n |X_n(\omega)|^p < \infty$ is equivalent to

$$\sup_{n\geq 1} |X_n(\omega)|^p < \infty$$

for all $\omega \in \Omega$. For any natural number $m \geq 1$, let us define

$$A_m = \left\{\omega \in \Omega : \sup_{n\geq 1} |X_n(\omega)|^p \leq m\right\}.$$

Assume that $r < p$ and fix $a > p/r - 1$. As $P(A_m) \to 1$ as $m \to \infty$, we can choose $m_1 \geq 1$ such that $P\big(A_{m_1}\big) > 1 - 2^{-a}$. Integrating and applying Fatou's lemma, we obtain

$$\sup_{n\geq 1} \int_{A_{m_1}} |X_n(\omega)|^p dP(\omega) \leq m_1. \tag{2}$$

We now apply the Biting Lemma (cf. [1]) to the sequence $(X_n)_{n\geq 1}$ and obtain: an increasing sequence of sets $(B^1_k)_{k\geq 1}$ in $\mathcal{F}$ with $P(B^1_k) \to 1$ as $k \to \infty$, and a subsequence $(X^1_n)_{n\geq 1}$ of $(X_n)_{n\geq 1}$ such that $(X^1_n)_{n\geq 1}$ is uniformly integrable on each set $A_{m_1} \cap B_k^1$, $k \geq 1$. The latter fact together with estimate (2) show that Theorem 0 applies to the sequence $(X_n^1)_{n\geq 1}$ and gives

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in A_{m_1} \cap B_k^1 : \left|\sum_{j=1}^{n} X_j^1(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0 \text{ and } k \geq 1.$$

Another application of the Biting Lemma to $(X_n^1)_{n\geq 1}$, instead of $(X_n)_{n\geq 1}$, produces: a measurable set A_{m_2} with $P\big(A_{m_2}\big) > 1-3^{-a}$, an increasing sequence of sets $(B_k^2)_{k\geq 1}$ in $\mathcal{F}$ with $P(B_k^2) \to 1$ as $k \to \infty$, and a subsequence $(X_n^2)_{n\geq 1}$ of $(X_n^1)_{n\geq 1}$ such that $(X_n^2)_{n\geq 1}$ is uniformly integrable on each set $A_{m_2} \cap B_k^2$, $k \geq 1$, such that

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in A_{m_2} \cap B_k^2 : \left|\sum_{j=1}^{n} X_j^2(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0 \text{ and } k \geq 1.$$

By induction, we construct for each $i \geq 1$: a measurable set A_{m_i} with $P\big(A_{m_i}\big) > 1-(i+1)^{-a}$, an increasing sequence of sets $(B_k^i)_{k\geq 1}$ in $\mathcal{F}$ with $P(B_k^i) \to 1$ as $k \to \infty$, and a subsequence $(X_n^i)_{n\geq 1}$ of $(X_n^{i-1})_{n\geq 1}$, with the convention that $(X_n^0)_{n\geq 1}$ is precisely $(X_n)_{n\geq 1}$, such that $(X_n^i)_{n\geq 1}$ is uniformly integrable on each set $A_{m_i} \cap B_k^i$, $k \geq 1$, and

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in A_{m_i} \cap B_k^i : \left|\sum_{j=1}^{n} X_j^i(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0 \text{ and } k, i \geq 1.$$

Now define $Y_n := X_n^n$ and, using a diagonal argument in the above formula, we obtain that

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in A_{m_n} \cap B_k^n : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0 \text{ and } k \geq 1. \quad (3)$$

As $P(B_k^n) \to 1$ as $k \to \infty$ for all $n \geq 1$, formula (3) and the dominated convergence theorem imply that

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in A_{m_n} : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0. \quad (4)$$

Therefore, to prove that series (1) converges for our subsequence $(Y_n)_{n\geq 1}$ and $r < p$, it suffices to prove (4) with A_{m_n} replaced by its complement, i.e.,

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in \Omega \setminus A_{m_n} : \left|\sum_{j=1}^{n} Y_j(\omega)\right| > \varepsilon n^{1/r}\right\}\right) < \infty \text{ for } \varepsilon > 0. \quad (5)$$

Indeed, the latter series is

$$\leq \sum_{n=1}^{\infty} n^{p/r-2} P\left(\left\{\omega \in \Omega \setminus A_{m_n}\right\}\right) \leq \sum_{n=1}^{\infty} n^{p/r-2-1} < \infty \quad (6)$$

as $P\big(A_{m_n}\big) > 1-(n+1)^{-a} > 1-n^{-a}$ and $a > p/r-1$. The proof is achieved in the case $r < p$.

If $r = p$, then we modify the induction process as follows: choose measurable sets A_{m_i} with $P(A_{m_i}) > i/(i+1)$ for all $i \geq 1$; as such, the diagonal argument above gives the following replacement of (4):

$$\sum_{n=1}^{\infty}\frac{1}{n}\bigg(\bigg\{\omega \in A_{m_n} : \bigg|\sum_{j=1}^{n} Y_j(\omega)\bigg| > \varepsilon n^{1/r}\bigg\}\bigg) < \infty \text{ for } \varepsilon > 0.$$

To show that series (1) converges for our subsequence $(Y_n)_{n\geq 1}$ and $r = p$, it suffices to prove the following replacement of (5):

$$\sum_{n=1}^{\infty}\frac{1}{n}P\bigg(\bigg\{\omega \in \Omega\setminus A_{m_n} : \bigg|\sum_{j=1}^{n} Y_j(\omega)\bigg| > \varepsilon n^{1/r}\bigg\}\bigg) < \infty \text{ for } \varepsilon > 0.$$

Indeed, the latter series is

$$\leq \sum_{n=1}^{\infty}\frac{1}{n}P\bigg(\bigg\{\omega \in \Omega\setminus A_{m_n}\bigg\}\bigg) \leq \sum_{n=1}^{\infty}\frac{1}{n(n+1)} < \infty$$

by the choice of $P(A_{m_n}), n \geq 1$. The latter is the substitute of (6) in the case $r = p$, and the proof is now complete.

Proof of Theorem 2. By hypothesis we can write

$$Z_n = \sum_{i\in I_n}\lambda_i^n|\tilde{X}_{n+i}|^p \text{ for some } \lambda_i^n \geq 0 \text{ with } \sum_{i\in I_n}\lambda_i^n = 1,$$

and where I_n are finite subsets of $\{0,1,2,\ldots\}$. In addition, the sequence $(Z_n)_{n\geq 1}$ satisfies the condition $\sup_{n\geq 1}|Z_n(\omega)|^p < \infty$ for all $\omega \in \Omega$. For any natural number $m \geq 1$, let us define $A_m = \Big\{\omega \in \Omega : \sup_{n\geq 1}|Z_n(\omega)| \leq m\Big\}$. As $P(A_m) \to 1$ as $m \to \infty$, we can choose $m_1 \geq 1$ such that $P\big(A_{m_1}\big) > 1-2^{-a}$ or $1/2$, according to $p > r$ or $p = r$, and where $a > p/r-1$ is fixed. Integrating and applying Fatou's lemma, we obtain

$$\sup_{n\geq 1}\sum_{i\in I_n}\lambda_i^n\int_{A_{m_1}}|\tilde{X}_{n+i}(\omega)|^p dP(\omega) \leq m_1.$$

Hence there is a subsequence $(\bar{X}_n)_{n\geq 1}$ of $(\tilde{X}_n)_{n\geq 1}$ (therefore of $(X_n)_{n\geq 1}$ as well), such that

$$\sup_{n\geq 1}\int_{A_{m_1}}|\bar{X}_n(\omega)|^p dP(\omega) \leq m_1,$$

which is precisely eq. (2) along a subsequence. The remainder of the proof goes exactly as in the proof of Theorem 1.

References

[1] 1 J.K. Brooks, R.V. Chacon, *Continuity and compactness of measures*, Adv. Math. 37 (1980) 16–26.

[2] C. Castaing, M. Saadoune, *Komlós type convergence for random variables and random sets with applications to minimization problems*, Adv. Math. Econ. 10 (2007) 1–29.

[3] S.J. Dilworth, *Convergence of series of scalar- and vector-valued random variables and a subsequence principle in L_2*, Trans. Amer. Math. Soc. 301 (1987) 375–384.

[4] E. Lesigne, D. Volný, *Large deviations for martingales*, Stoch. Proc. Appl. 96 (2001) 143–159.

[5] G. Stoica, *The Baum-Katz theorem for bounded subsequences*, Stat. Prob. Lett. 78 (2008), 924–926.

[6] H. von Weizsäcker, *Can one drop the L^1-boundedness in Komlós' subsequence theorem?* Amer. Math. Monthly 111 (2004) 900-903.

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