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Oscillation of Second Order Difference Equation with a Sub-linear Neutral Term

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Abstract: This paper deals with the oscillation of a certain class of second order difference equations with a sub-linear neutral term. Using some inequalities and Riccati type transformation, four new oscillation criteria are obtained. Examples are included to illustrate the main results.

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1. Introduction

In this paper, we are concerned with the oscillatory behavior of the nonlinear difference equation with a sub-linear neutral term

$$\Delta(a_n\Delta(x_n + p_n x_{n-k}^{\alpha})) + q_n x_{n+1-l}^{\beta} = 0,\ n \geq n_0, \tag{1.1}$$

where n_0 is a nonnegative integer, subject to the following conditions:

(H_1) $0 < \alpha \leq 1$ and β are ratios of odd positive integers;

(H_2) $\{a_n\}$, $\{p_n\}$, and $\{q_n\}$ are positive real sequences for all $n \geq n_0$;

(H_3) k is a positive integer, and l is a nonnegative integer.

Let $\theta = \max\{k, l\}$. By a *solution* of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ that satisfies equation (1.1) for all $n \geq n_0$. A solution of equation (1.1) is called *oscillatory* if its terms are neither eventually positive nor eventually negative, and *nonoscillatory* otherwise.

In the last few years there has been a great interest in investigating the oscillatory and asymptotic behavior of neutral type difference equations, see [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references cited therein.

In [4], Lin considered the equation of the form

$$\Delta(x_n - p_n x_{n-k}^{\alpha}) + q_n x_{n-l}^{\beta} = 0, \; n \geq n_0, \tag{1.2}$$

and studied its oscillatory behavior. In [5], Thandapani et al. investigated the oscillation of all solutions of the equation

$$\Delta(a_n \Delta(x_n - p x_{n-k}^{\alpha})) + q_n x_{n+1-l}^{\beta} = 0, \; n \geq n_0, \tag{1.3}$$

where $p > 0$ is a real number, k and l are positive integers, $0 < \alpha \leq 1$ and β are ratios of odd positive integers, and $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$.

A special case of the equation studied by Yildiz and Ogunmez [11] has the form

$$\Delta^2(x_n + p_n x_{n-k}^{\alpha}) + q_n x_{n-l}^{\beta} = 0, \tag{1.4}$$

where $\{p_n\}$ is a real sequence, $\{q_n\}$ is a nonnegative real sequence, and $\alpha > 1$ and $\beta > 0$ are again ratios of odd positive integers. They too discussed the oscillatory behavior of solutions.

In [6], Thandapani et al. considered equation (1.3), and obtained criteria for the oscillation of solutions provided $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$.

In this paper, we obtain sufficient conditions for the oscillation of all solutions of equation (1.1) in the two cases

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty \tag{1.5}$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty. \tag{1.6}$$

Our technique of proof makes use of some inequalities and Riccati type transformations. The results we obtain here are new and generalize those reported in [4, 5, 6, 11, 12]. Examples are provided to illustrate the main results.

2. Oscillation results

In this section, we obtain sufficient conditions for the oscillation of all solutions of equation (1.1). We set

$$z_n = x_n + p_n x_{n-k}^{\alpha}.$$

Due to the form of our equation, we only need to give proofs for the case of eventually positive nonoscillatory solutions since the proofs for eventually negative solutions would be similar.

We begin with the following two lemmas given in [7].

Lemma 2.1. *Assume that $\beta \geq 1$ and a, $b \in [0, \infty)$. Then*

$$a^{\beta} + b^{\beta} \geq \frac{1}{2^{\beta-1}}(a+b)^{\beta}.$$

Lemma 2.2. *Assume that* $0 < \beta \leq 1$ *and* $a,\, b \in [0, \infty)$. *Then*

$$a^\beta + b^\beta \geq (a+b)^\beta.$$

The next lemma can be found in [3, Theorem 41, p. 39].

Lemma 2.3. *Assume that* $a > 0$, $b > 0$, *and* $0 < \beta \leq 1$. *Then*

$$a^\beta - b^\beta \leq \beta b^{\beta-1}(a-b).$$

Here is our first oscillation result.

Theorem 2.4. *Assume that* (H_1)*–*(H_3) *and* (1.5) *hold. If* $\beta \geq 1$ *and there exists a positive nondecreasing real sequence* $\{\rho_n\}$ *such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left(\left[\frac{(1-\alpha p_{s+1-l})^\beta}{2^{\beta-1}} - \frac{(1-\alpha)^\beta p_{s+1-l}^\beta}{M^\beta} \right] \rho_s q_s - \frac{a_{s-l}(\Delta\rho_s)^2}{4\beta M^{\beta-1}\rho_s} \right) = \infty \quad (2.1)$$

holds for all constants $M > 0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution $\{x_n\}$, say $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. From equation (1.1), we have

$$\Delta(a_n \Delta z_n) = -q_n x_{n+1-l}^\beta < 0, \; n \geq n_1. \quad (2.2)$$

In view of condition (1.5), it is easy to see that $\Delta z_n > 0$ for all $n \geq n_1$. Now, it follows from the definition z_n, and using Lemma 2.3, we have

$$\begin{aligned} x_n &= z_n - p_n x_{n-k}^\alpha \geq z_n - p_n(z_n^\alpha - 1) - p_n \\ &\geq z_n - \alpha p_n(z_n - 1) - p_n \\ &= (1-\alpha p_n) z_n - (1-\alpha) p_n \end{aligned}$$

or

$$(x_{n+1-l} + (1-\alpha)p_{n+1-l})^\beta \geq (1-\alpha p_{n+1-l})^\beta z_{n+1-l}^\beta, \; n \geq n_1.$$

Using Lemma 2.1, in the last inequality, we obtain

$$x_{n+1-l}^\beta \geq \frac{1}{2^{\beta-1}}(1-\alpha p_{n+1-l})^\beta z_{n+1-l}^\beta - (1-\alpha)^\beta p_{n+1-l}^\beta, \; n \geq n_1. \quad (2.3)$$

From (2.2) and (2.3), we have

$$\Delta(a_n \Delta z_n) \leq \frac{-(1-\alpha p_{n+1-l})^\beta}{2^{\beta-1}} q_n z_{n+1-l}^\beta + (1-\alpha)^\beta q_n p_{n+1-l}^\beta, \; n \geq n_1. \quad (2.4)$$

Define

$$w_n = \frac{\rho_n a_n \Delta z_n}{z_{n-l}^\beta}, \; n \geq n_1. \quad (2.5)$$

Then, $w_n > 0$ for all $n \geq n_1$, and

$$\Delta w_n = \frac{\rho_n \Delta(a_n \Delta z_n)}{z^\beta_{n+1-l}} + \frac{(\Delta\rho_n)a_{n+1}\Delta z_{n+1}}{z^\beta_{n+1-l}} - \frac{\rho_n a_n \Delta z_n}{z^\beta_{n+1-l} z^\beta_{n-l}} \Delta(z^\beta_{n-l}). \tag{2.6}$$

By the Mean Value Theorem

$$z^\beta_{n+1-l} - z^\beta_{n-l} \geq \begin{cases} \beta z^\beta_{n-l}\Delta z_{n-l}, \ if\ \beta \geq 1, \\ \beta z^{\beta-1}_{n+1-l}\Delta z_{n-l}, \ if\ \beta < 1. \end{cases} \tag{2.7}$$

Combining (2.7) with (2.6) and then using the facts that $a_n \Delta z_n$ is positive and decreasing and z_n is increasing, we have

$$\begin{aligned} \Delta w_n \ \leq \ & \frac{-(1-\alpha p_{n+1-l})^\beta}{2^{\beta-1}} \rho_n q_n + \frac{\rho_n (1-\alpha)^\beta}{M^\beta} p^\beta_{n+1-l}\rho_n q_n \\ & + \frac{\Delta\rho_n w_{n+1}}{\rho_{n+1}} - \beta M^{\beta-1} \frac{\rho_n}{\rho^2_{n+1} a_{n-l}} w^2_{n+1}, \ n \geq n_1, \end{aligned} \tag{2.8}$$

where we have used the fact that $z_n \geq M$ for some $M > 0$ and all $n \geq n_1$. Completing the square on the last two terms on the right, we obtain

$$\Delta w_n \ \leq \ -\left[\frac{(1-\alpha p_{n+1-l})^\beta}{2^{\beta-1}} - \frac{(1-\alpha)^\beta}{M^\beta} p^\beta_{n+1-l}\right] \rho_n q_n + \frac{a_{n-l}(\Delta\rho_n)^2}{4\beta M^{\beta-1}\rho_n}, \ n \geq n_1.$$

Summing the last inequality from n_1 to n yields

$$\sum_{s=n_1}^{n} \left(\left[\frac{(1-\alpha p_{s+1-l})^\beta}{2^{\beta-1}} - \frac{(1-\alpha)^\beta}{M^\beta} p^\beta_{s+1-l}\right] \rho_s q_s - \frac{a_{s-l}(\Delta\rho_s)^2}{4\beta M^{\beta-1}\rho_s} \right) \leq w_{n_1},$$

which contradicts (2.1) and completes the proof of the theorem. □

The proof of the following theorem is similar to that of Theorem 2.4 only using Lemma 2.2 instead of Lemma 2.1. We omit the details.

Theorem 2.5. *Assume that* (H_1)–(H_3) *and* (1.5) *hold. If* $0 < \beta < 1$ *and there exists a positive nondecreasing real sequence* $\{\rho_n\}$ *such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left(\left[(1-\alpha p_{s+1-l})^\beta - \frac{(1-\alpha)^\beta}{M^\beta} p^\beta_{s+1-l}\right] \rho_s q_s - \frac{a_{s-l}(\Delta\rho_s)^2}{4\beta M^{\beta-1}\rho_s} \right) = \infty \tag{2.9}$$

holds for all constants $M > 0$, *then every solution of equation* (1.1) *is oscillatory.*

Our next two theorems are for the case where (1.6) holds in place of (1.5). We let

$$A_n = \sum_{s=n}^{\infty} \frac{1}{a_s}.$$

We will also need the condition

$$1 - \alpha p_n \frac{A_{n-k}}{A_n} > 0 \text{ for all } n \geq n_0. \tag{2.10}$$

Theorem 2.6. *Let $\beta \geq 1$ and (H_1)–(H_3), (1.6), and (2.10) hold. Assume that there exists a positive nondecreasing real sequence $\{\rho_n\}$ such that (2.1) holds for all constants $M > 0$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left(A_{s+1}^{\beta} \left[\left(1 - \alpha p_{s+1-l} \frac{A_{s+1-l-k}}{A_{s+1-l}}\right)^{\beta} \frac{1}{2^{\beta-1}} - \frac{(1-\alpha)^{\beta} p_{s+1-l}^{\beta}}{D^{\beta} A_{s+1}^{\beta}} \right] q_s - \frac{\beta A_s^{\beta-1}}{4D^{\beta-1} a_s A_{s+1}^{\beta}} \right) = \infty \tag{2.11}$$

holds for every constant $D > 0$, then every solution of equation (1.1) is oscillatory.

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution such that $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1 \geq n_0$. From (1.1), we have that (2.2) holds. We then have that either $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n > 0$ holds, then we can proceed as in the proof of Theorem 2.4 and again obtain a contradiction to (2.1).

Now assume that $\Delta z_n < 0$ for all $n \geq n_1$. Define

$$u_n = \frac{a_n \Delta z_n}{z_n^{\beta}}, \quad n \geq n_1. \tag{2.12}$$

Then $u_n < 0$ for all $n \geq n_1$ and from (2.2), we have

$$\Delta z_s \leq \frac{a_n \Delta z_n}{a_s}, \quad s \geq n.$$

Summing the last inequality from n to j, we obtain

$$z_{j+1} - z_n \leq a_n \Delta z_n \sum_{s=n}^{j} \frac{1}{a_s};$$

and then letting $j \to \infty$ gives

$$\frac{a_n \Delta z_n A_n}{z_n} \geq -1, \quad n \geq n_1. \tag{2.13}$$

Thus,

$$\frac{-a_n \Delta z_n (-a_n \Delta z_n)^{\beta-1} A_n^{\beta}}{z_n^{\beta}} \leq 1$$

for $n \geq n_1$. Since $-a_n \Delta z_n > 0$ and (2.2) and (2.12) hold, we have

$$-\frac{1}{L^{\beta-1}} \leq u_n A_n^{\beta} \leq 0, \tag{2.14}$$

where $L = -a_{n_1} \Delta z_{n_1}$. On the other hand, from (2.13),

$$\Delta \left(\frac{z_n}{A_n} \right) \geq 0, \quad n \geq n_1. \tag{2.15}$$

From the definition of z_n, (2.15), and Lemma 2.3, we have

$$\begin{aligned} x_n &= z_n - p_n x_{n-k}^{\alpha} \geq z_n - p_n(z_{n-k}^{\alpha} - 1) - p_n \\ &\geq z_n - \alpha p_n(z_{n-k} - 1) - p_n \\ &\geq \left(1 - \alpha p_n \frac{A_{n-k}}{A_n}\right) z_n + (\alpha - 1)p_n, \end{aligned}$$

or

$$(x_{n+1-l} + (1-\alpha)p_{n+1-l})^{\beta} \geq \left(1 - \alpha p_{n+1-l} \frac{A_{n+1-l-k}}{A_{n+1-l}}\right)^{\beta} z_{n+1-l}^{\beta}.$$

Using Lemma 2.1, in the last inequality, we obtain

$$x_{n+1-l}^{\beta} \geq \frac{1}{2^{\beta-1}} \left(1 - \alpha p_{n+1-l} \frac{A_{n+1-l-k}}{A_{n+1-l}}\right)^{\beta} z_{n+1-l}^{\beta} - (1-\alpha)^{\beta} p_{n+1-l}^{\beta}. \tag{2.16}$$

From (2.2) and (2.16), we have

$$\Delta(a_n \Delta z_n) \leq -\frac{q_n}{2^{\beta-1}} \left(1 - \alpha p_{n+1-l} \frac{A_{n+1-l-k}}{A_{n+1-l}}\right)^{\beta} z_{n+1-l}^{\beta} + q_n (1-\alpha)^{\beta} p_{n+1-l}^{\beta}. \tag{2.17}$$

From (2.12),

$$\Delta u_n = \frac{\Delta(a_n \Delta z_n)}{z_{n+1}^{\beta}} - \frac{a_n \Delta z_n}{z_n^{\beta} z_{n+1}^{\beta}} \Delta z_n^{\beta}, \ n \geq n_1. \tag{2.18}$$

By the Mean Value Theorem,

$$z_{n+1}^{\beta} - z_n^{\beta} \leq \begin{cases} \beta z_{n+1}^{\beta-1} \Delta z_n, \ if \ \beta \geq 1, \\ \beta z_n^{\beta-1} \Delta z_n, \ if \ 0 < \beta < 1, \end{cases} \tag{2.19}$$

so combining (2.19) and (2.18) and using the fact that $\Delta z_n < 0$ gives

$$\Delta u_n \leq \frac{\Delta(a_n \Delta z_n)}{z_{n+1}^{\beta}} - \beta \frac{u_n^2}{a_n} z_n^{\beta-1}. \tag{2.20}$$

Since z_n/A_n is increasing, there is a constant $D > 0$ such that $z_n/A_n \geq D$ for $n \geq n_1$. Using this together with (2.15) and (2.17) in (2.20), we obtain

$$\Delta u_n \leq \frac{-q_n}{2^{\beta-1}} \left(1 - \alpha p_{n+1-l} \frac{A_{n+1-l-k}}{A_{n+1-l}}\right)^{\beta} + \frac{q_n(1-\alpha)^{\beta}}{D^{\beta} A_{n+1}^{\beta}} p_{n+1-l}^{\beta} - \beta D^{\beta-1} A_n^{\beta-1} \frac{u_n^2}{a_n}. \tag{2.21}$$

Multiplying (2.21) by A_{n+1}^{β} and then summing the resulting inequality from n_1 to $n-1$, we see that

$$\begin{aligned} A_n^{\beta} u_n - A_{n_1}^{\beta} u_{n_1} + \sum_{s=n_1}^{n-1} A_{s+1}^{\beta} \left[\left(1 - \alpha p_{s+1-l} \frac{A_{s+1-l-k}}{A_{s+1-l}}\right)^{\beta} \frac{1}{2^{\beta-1}} - \frac{(1-\alpha)^{\beta}}{D^{\beta} A_{s+1}^{\beta}} p_{s+1-l}^{\beta} \right] q_s \\ + \sum_{s=n_1}^{n-1} \frac{\beta A_s^{\beta-1} u_s}{a_s} + \sum_{s=n_1}^{n-1} \beta D^{\beta-1} A_s^{\beta-1} A_{s+1}^{\beta} \frac{u_s^2}{a_s} \leq 0, \end{aligned}$$

which upon completing the square on the last two terms yields

$$\sum_{s=n_1}^{n-1} \left(A_{s+1}^{\beta} \left[\left(1 - \alpha p_{s+1-l} \frac{A_{s+1-l-k}}{A_{s+1-l}} \right)^{\beta} \frac{1}{2^{\beta-1}} - \frac{(1-\alpha)^{\beta}}{D^{\beta} A_{s+1}^{\beta}} p_{s+1-l}^{\beta} \right] q_s \right. \\ \left. - \frac{\beta A_s^{\beta-1}}{4D^{\beta-1} a_s A_{s+1}^{\beta}} \right) \le \frac{1}{L^{\beta-1}} + A_{n_1} u_{n_1}$$

in view of (2.14). This contradicts (2.11), and completes the proof of the theorem. □

The proof of the following theorem is similar to that of Theorem 2.6 using Lemma 2.2 instead of Lemma 2.1. We again omit the details.

Theorem 2.7. *Let* $0 < \beta < 1$ *and* (H_1)–(H_3), (1.6), *and* (2.10) *hold. Assume that there exists a positive nondecreasing real sequence* $\{\rho_n\}$ *such that* (2.9) *holds for all constants* $M > 0$. *If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left(A_{s+1}^{\beta} \left[\left(1 - \alpha p_{s+1-l} \frac{A_{s+1-l-k}}{A_{s+1-l}} \right)^{\beta} - \frac{(1-\alpha)^{\beta} p_{s+1-l}^{\beta}}{D^{\beta} A_{s+1}^{\beta}} \right] q_s \right. \\ \left. - \frac{\beta A_s^{\beta-1}}{4D^{\beta-1} a_s A_{s+1}^{\beta}} \right) = \infty \quad (2.22)$$

holds for all constants $D > 0$, *then every solution of equation* (1.1) *is oscillatory.*

3. Examples

In this section, we present two examples to illustrate our main results.

Example 3.1. Consider the neutral difference equation

$$\Delta\left((n+1)\Delta\left(x_n + \frac{1}{n} x_{n-2}^{1/3}\right)\right) + \left(4n + 10 + \frac{2n+1}{n(n+1)}\right) x_{n-3}^3 = 0, \; n \ge 1. \quad (3.1)$$

Here $a_n = (n+1)$, $p_n = \frac{1}{n}$, $q_n = 4n + 10 + \frac{2n+1}{n(n+1)}$, $\alpha = \frac{1}{3}$, $\beta = 3$, $k = 2$, and $l = 4$. By taking $\rho_n = 1$, we see that all conditions of Theorem 2.4 are satisfied and hence every solution of equation (3.1) is oscillatory. In fact $\{x_n\} = \{(-1)^{3n}\}$ is one such oscillatory solution of equation (3.1).

Example 3.2. Consider the neutral difference equation

$$\Delta\left((n+1)(n+2)\Delta\left(x_n + \frac{1}{n(n+1)} x_{n-1}^{1/3}\right)\right) \\ + \left(4(n+2)^2 - \frac{2(2n^2+4n+1)}{n(n+1)}\right) x_{n-1}^3 = 0, \; n \ge 1. \quad (3.2)$$

Here $a_n = (n+1)(n+2)$, $p_n = \frac{1}{n(n+1)}$, $q_n = 4(n+2)^2 - \frac{2(2n^2+4n+1)}{n(n+1)}$, $\alpha = \frac{1}{3}$, $\beta = 3$, $k = 1$, and $l = 2$. Simple calculation shows that $A_n = \frac{1}{n+1}$ and $1 - \alpha p_n \frac{A_{n-k}}{A_n} = 1 - \frac{1}{3n^2} > 0$. The conditions (2.1) and (2.11) are also satisfied with $\rho_n = 1$. Therefore, by Theorem 2.6, every solution of equation (3.2) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (3.2).

We conclude this paper with the following remark.

Remark 3.3. Condition (2.10) is somewhat restrictive. It implies that we must have $\{p_n\} \to 0$ as $n \to \infty$. It would be good to see a result that did not need this added condition. Note also that it can be seen from the proof of Theorem 2.6 that (2.10) is not needed if $\alpha = 1$.

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