De la Vallée Poussin Summability, the Combinatorial Sum $\sum_{k=n}^{2n-1} \binom{2k}{k}$ and the de la Vallée Poussin Means Expansion

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Abstract: In this paper we apply the de la Vallée Poussin sum to a combinatorial Chebyshev sum by Ziad S. Ali in [1]. One outcome of this consideration is the main lemma proving the following combinatorial identity: with $Re(z)$ standing for the real part of z we have

$$\sum_{k=n}^{2n-1} \binom{2k}{k} = Re\left(\binom{2n}{n} {}_2F_1(1, 1/2+n; 1+n; 4)\right.$$

$$\left. - \binom{4n}{2n} {}_2F_1(1, 1/2+2n; 1+2n; 4)\right).$$

Our main lemma will indicate in its proof that the hypergeometric factors

$${}_2F_1(1, 1/2+n; 1+n; 4), \quad \text{and} \quad {}_2F_1(1, 1/2+2n; 1+2n; 4)$$

are complex, each having a real and imaginary part.

As we apply the de la Vallée Poussin sum to the combinatorial Chebyshev sum generated in the Key lemma by Ziad S. Ali in [1], we see in the proof of the main lemma the extreme importance of the use of the main properties of the gamma function. This represents a second important consideration.

A third new outcome are two interesting identities of the hypergeometric type with their new Meijer G function analogues. A fourth outcome is that by the use of the Cauchy integral formula for the derivatives we are able to give a different meaning to the sum:

$$\sum_{k=n}^{2n-1} \binom{2k}{k}.$$

A fifth outcome is that by the use of the Gauss-Kummer formula we are able to make better sense of the expressions

$$\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4),\quad \text{and}\quad \binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)$$

by making use of the series definition of the hypergeometric function. As we continue we notice a new close relation of the Key lemma, and the de la Vallée Poussin means. With this close relation we were able to talk about the de la Vallée Poussin summability of the two infinite series $\sum_{n=0}^{\infty}\cos n\theta$, and $\sum_{n=0}^{\infty}(-1)^n\cos n\theta$.

Furthermore the application of the de la Vallée Poussin sum to the Key lemma has created two new expansions representing the following functions:

$$\frac{2^{(n-1)}(1+x)^n(-1+2^n(1+x)^n)}{n(2x+1)},\quad \text{where}\quad x=\cos\theta,$$

and

$$\frac{-2^{(n-1)}(-1+2^n(1-x)^n)(1-x)^n}{n(2x-1)},\quad \text{where}\quad x=\cos\theta$$

in terms of the de la Vallée Poussin means of the two infinite series

$$\sum_{n=0}^{\infty}\cos n\theta\ ,$$

and

$$\sum_{n=0}^{\infty}(-1)^n\cos n\theta\ .$$