Weak Solutions of Fractional Order
Differential Equations via Volterra-Stieltjes
Integral Operator

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Abstract: The fractional derivative of the Riemann-Liouville and Caputo types played an important role in the development of the theory of fractional derivatives, integrals and for its applications in pure mathematics ([18], [21]). In this paper, we study the existence of weak solutions for fractional differential equations of Riemann-Liouville and Caputo types. We depend on converting of the mentioned equations to the form of functional integral equations of Volterra-Stieltjes type in reflexive Banach spaces.

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1. Introduction and preliminaries

Let $E$ be a reflexive Banach space with norm $\| \cdot \|$ and dual $E^*$. Denote by $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm. Fractional differential equations have received increasing attention due to its applications in physics, chemistry, materials, engineering, biology, finance [15, 16]. Fractional order derivatives have the memory property and can describe many phenomena that integer order derivatives cant characterize. Only a few papers consider fractional differential equations in reflexive Banach spaces with the weak topology [6, 7, 14, 22, 23].

Here we study the existence of weak solutions of the Volterra-Stieltjes integral equation

$$x(t) = p(t) + \int_0^t f(s, x(s)) \, ds \, g(t, s), \quad t \in I = [0, T],$$
in the reflexive Banach space $E$.

Let $\alpha \in (0,1)$. As applications, we study the existence of weak solution for the differential equations of fractional order

$$RD^\alpha x(t) = f(t,x(t)), \ t \in (0,T)$$

(1.1)

with the initial data

$$x(0) = 0,$$  \hspace{1cm} (1.2)

where $RD^\alpha x(.)$ is a Riemann-Liouville fractional derivative of the function $x : I = [0,T] \to E$.

Also we study the existence of mild solution for the initial value problem

$$CD^\alpha x(t) = f(t,x(t)), \ t \in (0,T)$$

(1.3)

with the initial data

$$x(0) = x_0,$$  \hspace{1cm} (1.4)

where $CD^\alpha x(.)$ is a Caputo fractional derivative of the function $x : I = [0,T] \to E$.

Functional integral equations of Volterra-Stieltjes type have been studied in the space of continuous functions in many papers for example, (see [1-5] and [8]). For the properties of the Stieltjes integral (see Banas [1]).

Definition 1.1. The fractional (arbitrary) order integral of the function $f \in L_1$ of order $\alpha > 0$ is defined as [18, 21]

$$I^\alpha f(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds.$$  

For the fractional-order derivative we have the following two definitions.

Definition 1.2. The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0,1)$ is defined as ([18], [21])

$$RD^\alpha_a f(t) = \frac{d}{dt} \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s) \, ds$$

or

$$RD^\alpha_a f(t) = \frac{d}{dt} t_1^{1-\alpha} f(t).$$

Definition 1.3. The Caputo fractional-order derivative of $g(t)$ of order $\alpha \in (0,1]$ of the absolutely continuous function $g(t)$ is defined as ([9])

$$CD^\alpha_a g(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} g(s) \, ds$$

or

$$CD^\alpha_a g(t) = t_1^{1-\alpha} \frac{d}{dt} g(t).$$
Now, we shall present some auxiliary results that will be need in this work. Let $E$ be a Banach space (need not be reflexive) and let $x : [a, b] \rightarrow E$, then

1. $x(\cdot)$ is said to be weakly continuous (measurable) at $t_0 \in [a, b]$ if for every $\phi \in E^*$, $\phi(x(\cdot))$ is continuous (measurable) at $t_0$.

2. A function $h : E \rightarrow E$ is said to be weakly sequentially continuous if $h$ maps weakly convergent sequences in $E$ to weakly convergent sequences in $E$.

If $x$ is weakly continuous on $I$, then $x$ is strongly measurable and hence weakly measurable (see [10] and [13]). It is evident that in reflexive Banach spaces, if $x$ is weakly continuous function on $[a, b]$, then $x$ is weakly Riemann integrable (see [13]).

**Definition 1.4.** Let $f : I \times E \rightarrow E$. Then $f(t, u)$ is said to be weakly-weakly continuous at $(t_0, u_0)$ if given $\epsilon > 0$, $\phi \in E^*$ there exists $\delta > 0$ and a weakly open set $U$ containing $u_0$ such that

$$| \phi(f(t, u) - f(t_0, u_0)) | < \epsilon$$

whenever $| t - t_0 | < \delta$ and $u \in U$.

Now, we have the following fixed point theorem, due to O’Regan, in the reflexive Banach space (see [19]) and some propositions which will be used in the sequel [13, 20].

**Theorem 1.5.** Let $E$ be a Banach space and let $Q$ be a nonempty, bounded, closed and convex subset of $C[I, E]$ and let $F : Q \rightarrow Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in $E$ for each $t \in I$. Then, $F$ has a fixed point in the set $Q$.

**Proposition 1.6.** A convex subset of a normed space $E$ is closed if and only if it is weakly closed.

**Proposition 1.7.** A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.

**Proposition 1.8.** Let $E$ be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\phi \in E^*$ with $\| \phi \| = 1$ and $\| y \| = \phi(y)$.

2. Volterra-Stieltjes integral equation

In this section we prove the existence of weak solutions for the Volterra-Stieltjes integral equation

$$x(t) = p(t) + \int_0^t f(s, x(s)) \, ds g(t, s), \quad t \in I = [0, T], \quad (2.5)$$
in the space $C[I,E]$. To facilitate our discussion, denote $\Lambda$ by

$$\Lambda = \{(t,s) : 0 \leq s \leq t \leq T\}$$

and let $p : I \to E$, $f : I \times E \to E$ and $g : \Lambda \to R$ be functions such that:

(i) $p \in C[I,E]$.

(ii) The function $f$ is weakly-weakly continuous.

(iii) There exists a constant $M$ such that $\|f(t,x)\| \leq M$.

(iv) The function $g$ is continuous on $\Lambda$.

(v) The function $s \to g(t,s)$ is of bounded variation on $[0,t]$ for each fixed $t \in I$.

(vi) For any $\epsilon > 0$ there exists $\delta > 0$ for all $t_1, t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \delta$ the following inequality holds

$$\int_{0}^{t_1} [g(t_2,s) - g(t_1,s)] \leq \epsilon.$$ 

(vii) $g(t,0) = 0$ for any $t \in I$.

Obviously we will assume that $g$ satisfies assumptions (iv)-(vi). For our purposes we will only need the following lemmas.

**Lemma 2.1.** [5] The function $z \to \bigvee_{z=0}^{t} g(t,s)$ is continuous on $[0,t]$ for any fixed $t \in I$.

**Lemma 2.2.** [5] For an arbitrary fixed $0 < t_2 \in I$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that if $t_1 \in I$, $t_1 < t_2$ and $t_2 - t_1 \leq \delta$ then

$$\bigvee_{s=t_1}^{t_2} g(t_2,s) \leq \epsilon.$$

**Lemma 2.3.** [5] The function $t \to \bigvee_{s=0}^{t} g(t,s)$ is continuous on $I$. Then there exists a finite positive constant $K$ such that

$$K = \sup \{ \bigvee_{s=0}^{t} g(t,s) : t \in I \}.$$ 

**Definition 2.4.** By a weak solution to (2.5) we mean a function $x \in C[I,E]$ which satisfies the integral equation (2.5). This is equivalent to find $x \in C[I,E]$ with

$$\phi(x(t)) = \phi(p(t) + \int_{0}^{t} f(s,x(s)) \, ds \, g(t,s)), \quad t \in I \forall \phi \in E^*.$$
Now we can prove the following theorem.

**Theorem 2.5.** Under the assumptions (i)-(vii), the Volterra-Stieltjes integral equation (2.5) has at least one weak solution $x \in C[I, E]$.

**Proof.** Define the nonlinear Volterra-Stieltjes integral operator $A$ by

$$Ax(t) = p(t) + \int_0^t f(s, x(s)) \, ds, \quad t \in I.$$ 

For every $x \in C[I, E]$, $f(., x(\cdot))$ is weakly continuous ([24]). To see this we equip $E$ and $I \times E$ with weak topology and note that $t \mapsto (t, x(t))$ is continuous as a mapping from $I$ into $I \times E$, then $f(\cdot, x(\cdot))$ is a composition of this mapping with $f$ and thus for each weakly continuous $x : I \to E$, $f(\cdot, x(\cdot)) : I \to E$ is weakly continuous, means that $\phi(f(\cdot, x(\cdot)))$ is continuous, for every $\phi \in E^*$, $g$ is of bounded variation. Hence $f(\cdot, x(\cdot))$ is weakly Riemann-Stieltjes integrable on $I$ with respect to $s \to g(t, s)$. Thus $A$ makes sense.

For notational purposes $\|x\|_0 = \sup_{t \in I} \|x(t)\|$.

Now, define the set $Q$ by

$$Q = \left\{ x \in C[I, E] : \| x \|_0 \leq M_0, \right\}$$

$$\| x(t_2) - x(t_1) \| \leq \| p(t_2) - p(t_1) \| + MN(\epsilon) + M \int_{s=t_1}^{t_2} g(t_2, s)$$

First notice that $Q$ is convex and norm closed. Hence $Q$ is weakly closed by Proposition 1.6.

Note that $A$ is well defined, to see that, Let $t_1, t_2 \in I$, $t_2 > t_1$, without loss of generality, assume $Ax(t_2) - Ax(t_1) \neq 0$.

$$\| Ax(t_2) - Ax(t_1) \| = \phi(Ax(t_2) - Ax(t_1)) \leq |\phi(p(t_2) - p(t_1))|$$

$$+ | \int_0^{t_2} \phi(f(s, x(s))) \, ds - \int_0^{t_1} \phi(f(s, x(s))) \, ds |$$

$$\leq |\| p(t_2) - p(t_1) \| + | \int_0^{t_1} \phi(f(s, x(s))) \, ds |$$

$$+ \int_{t_1}^{t_2} \phi(f(s, x(s))) \, ds |$$

$$\leq |\| p(t_2) - p(t_1) \| + | \int_0^{t_1} \phi(f(s, x(s))) \, ds |$$

$$+ \int_{t_1}^{t_2} \phi(f(s, x(s))) \, ds |$$

$$\leq | \int_0^{t_2} \phi(f(s, x(s))) \, ds - g(t_2, s) - g(t_1, s) |$$

$$+ \int_{t_1}^{t_2} \phi(f(s, x(s))) \, ds |$$
\[ \begin{align*}
&\leq \| p(t_2) - p(t_1) \| \\
&+ \int_0^{t_1} | \phi(f(s, x(s))) | \, ds \left[ \sum_{z=0}^{s} (g(t_2, z) - g(t_1, z)) \right] \\
&+ \int_{t_1}^{t_2} | \phi(f(s, x(s))) | \, ds \left[ \sum_{z=0}^{s} g(t_2, z) \right] \\
&\leq \| p(t_2) - p(t_1) \| + M \int_0^{t_1} ds \left[ \sum_{z=0}^{s} (g(t_2, z) - g(t_1, z)) \right] \\
&+ M \int_{t_1}^{t_2} ds \left[ \sum_{z=0}^{s} g(t_2, z) \right] \\
&\leq \| p(t_2) - p(t_1) \| + M \int_0^{t_1} ds \left[ \sum_{z=0}^{s} (g(t_2, s) - g(t_1, s)) \right] \\
&+ M \int_{t_1}^{t_2} ds \left[ \sum_{z=0}^{s} g(t_2, s) \right] \\
&\leq \| p(t_2) - p(t_1) \| + MN(\epsilon) + M \int_{s=t_1}^{t_2} g(t_2, s) ,
\end{align*} \]

where

\[ N(\epsilon) = \sup \left\{ \int_{s=0}^{t_1} (g(t_2, s) - g(t_1, s)) : t_1, t_2 \in I, t_1 < t_2, t_2 - t_1 \leq \epsilon \right\} . \]

Hence

\[ \| Ax(t_2) - Ax(t_1) \| \leq \| p(t_2) - p(t_1) \| + MN(\epsilon) + M \int_{s=t_1}^{t_2} g(t_2, s) , \quad (2.6) \]

and so \( Ax \in C[I, E] \). We claim that \( A : Q \to Q \) is weakly sequentially continuous and \( A(Q) \) is weakly relatively compact. Once the claim is established, Theorem 1.5 guarantees the existence of a fixed point \( x \in C[I, E] \) of the operator \( A \) and the integral equation (2.5) has a solution \( x \in C[I, E] \).

To prove our claim, we start by showing that \( A : Q \to Q \). Take \( x \in Q \), note that the inequality (2.6) shows that \( AQ \) is norm continuous. Then by using Proposition 1.8
we get

\[ \| Ax(t) \| = \phi(Ax(t)) \leq \| p(t) \| + | \phi(\int_0^t f(s, x(s)) \, ds g(t, s)) | \]

\[ \leq \| p(t) \| + \int_0^t | \phi(f(s, x(s))) | \, ds(\Delta g(t, z)) \]

\[ \leq \| p(t) \| + M \int_0^t ds(\Delta g(t, z)) \]

\[ \leq \| p(t) \| + M \sup_{t \in I} \int_0^t g(t, s) \]

\[ \leq \| p \|_0 + M \sup_{t \in I} \int_0^t g(t, s) \]

\[ \leq \| p \|_0 + MK = M_0. \]

Then

\[ \| Ax \|_0 = \sup_{t \in I} \| Ax(t) \| \leq M_0. \]

Hence, \( Ax \in Q \) and \( AQ \subset Q \) which prove that \( A : Q \to Q \), and \( AQ \) is bounded in \( C[I, E] \).

We need to prove now that \( A : Q \to Q \) is weakly sequentially continuous. Let \( \{x_n(t)\} \) be sequence in \( Q \) weakly convergent to \( x(t) \) in \( E \), since \( Q \) is closed we have \( x \in Q \). Fix \( t \in I \), since \( f \) satisfies (ii), then we have \( f(t, x_n(t)) \) converges weakly to \( f(t, x(t)) \). By the Lebesgue dominated convergence theorem (see assumption (iii)) for Pettis integral ([12]), we have for each \( \phi \in E^* \), \( s \in I \)

\[ \phi(\int_0^t f(s, x_n(s)) \, ds g(t, s)) = \int_0^t \phi(f(s, x_n(s))) \, ds g(t, s) \]

\[ \rightarrow \int_0^t \phi(f(s, x(s))) \, ds g(t, s), \forall \phi \in E^*, t \in I, \]

i.e. \( \phi(Ax_n(t)) \to \phi(Ax(t)) \), \( \forall t \in I \), \( Ax_n(t) \) converging weakly to \( Ax(t) \) in \( E \).

Thus, \( A \) is weakly sequentially continuous on \( Q \).

Next we show that \( AQ(t) \) is relatively weakly compact in \( E \).

Note that \( Q \) is nonempty, closed, convex and uniformly bounded subset of \( C[I, E] \)
and \( AQ \) is bounded in norm. According to Propositions 1.6 and 1.7, \( AQ \) is relatively weakly compact in \( C[I, E] \) implies \( AQ(t) \) is relatively weakly compact in \( E \), for each \( t \in I \).

Since all conditions of Theorem 1.5 are satisfied, then the operator \( A \) has at least one fixed point \( x \in Q \) and the nonlinear Stieltjes integral equation (2.5) has at least one weak solution \( x \in C[I, E] \).
3. Volterra integral equation of fractional order

In this section we show that the Volterra integral equation of fractional order

\[ x(t) = p(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \, ds, \quad t \in I \]  

(3.7)
can be considered as a special case of the Volterra-Stieltjes integral equation (2.1), where the integral is in the sense of weakly Riemann.

First, consider, as previously, that the function \( g(t,s) = g : \Lambda \rightarrow R \). Moreover, we will assume that the function \( g \) satisfies the following condition

(vi') For \( t_1, t_2 \in I, \ t_1 < t_2 \), the function \( s \to g(t_2,s) - g(t_1,s) \) is nonincreasing on \([0,t_1]\).

Now, we have the following lemmas which proved by Banaś et al. [5].

Lemma 3.1. Under assumptions (vi’) and (vii), for any fixed \( s \in I \), the function \( t \to g(t,s) \) is nonincreasing on \([s,1]\).

Lemma 3.2. Under assumptions (iv), (vi’) and (vii), the function \( g \) satisfies assumption (vi).

Consider the function \( g \) defined by

\[ g(t,s) = \frac{t^\alpha - (t-s)^\alpha}{\Gamma(\alpha + 1)}. \]  

(3.8)

Now, we show that the function \( g \) satisfies assumptions (iv), (v), (vi’) and (vii). Also we get

\[ d_s g(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} > 0 \]

for \( 0 \leq s < t \). This implies that \( s \to g(t,s) \) is increasing on \([0,t]\) for any fixed \( t \in I \). Thus the function \( g \) satisfies assumption (v).

To show that \( g \) satisfies assumption (vi’), let us fix arbitrary \( t_1,t_2 \in [0,T], \ t_1 < t_2 \). Then we get

\[ G(s) = g(t_2,s) - g(t_1,s) = \frac{t_2^\alpha - t_1^\alpha - (t_2-s)^\alpha + (t_1-s)^\alpha}{\Gamma(\alpha + 1)}, \]

define on \([0,t_1]\). Thus

\[ G'(s) = \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)} \left( \frac{1}{(t_2-s)^{1-\alpha}} - \frac{1}{(t_1-s)^{1-\alpha}} \right). \]

Hence \( G'(s) < 0 \) for \( s \in [0,t_1] \). This means that \( g \) satisfies assumption (vi’). And the function \( g \) satisfies assumptions (iv)-(vii) in Theorem 2.5.

Hence, the equation (3.7) can be written in the form

\[ x(t) = p(t) + \int_0^t f(s, x(s)) \, ds \, g(t,s). \]
And the equation (3.7) is a special case of the equation (2.5).
Now, we estimate the constants $K$, $N(\epsilon)$ used in our proof. To see this, since the function $s \to g(t, s)$ is nondecreasing on $[0, t]$ for any fixed $t \in I$. Then we have

$$\int_{s=0}^{t} g(t, s) = g(t, t) - g(t, 0) = g(t, t) = \frac{t^\alpha}{\Gamma(\alpha + 1)},$$

and

$$\int_{s=0}^{t_1} (g(t_2, s) - g(t_1, s)) = \sum_{i=1}^{n} [g(t_2, s_i) - g(t_1, s_i)] - [g(t_2, s_{i-1}) - g(t_1, s_{i-1})]$$

$$= \sum_{i=1}^{n} [g(t_2, s_{i-1}) - g(t_1, s_{i-1})] - [g(t_2, s_i) - g(t_1, s_i)]$$

$$= g(t_1, t_1) - g(t_2, t_1)$$

$$= \frac{1}{\Gamma(\alpha + 1)}[t_1^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha].$$

Thus

$$K = \sup\{ \int_{s=0}^{t} g(t, s) : t \in I \} = \frac{T^\alpha}{\Gamma(\alpha + 1)}$$

and

$$N(\epsilon) = \sup\{ \int_{s=0}^{t_1} (g(t_2, s) - g(t_1, s)) : t_1, t_2 \in I, t_1 < t_2, t_2 - t_1 \leq \epsilon \}$$

$$= \frac{1}{\Gamma(\alpha + 1)}[t_1^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha].$$

Since

$$\int_{s=t_1}^{t_2} g(t_2, s) = g(t_2, t_2) - g(t_2, t_1)$$

$$= \frac{1}{\Gamma(\alpha + 1)}[t_2^\alpha - (t_2 - t_2)^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha]$$

$$= \frac{(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)},$$

Then

$$Q = \{ x \in C[I, E] : \| x \|_0 \leq M_0 ,$$

$$\| x(t_2) - x(t_1) \| \leq \| p(t_2) - p(t_1) \| + \frac{M}{\Gamma(\alpha + 1)}[t_1^\alpha - t_2^\alpha + 2(t_2 - t_1)^\alpha] \}.$$ 

Finally, we can formulate the following existence result concerning the fractional integral equation (3.7).

**Theorem 3.3.** Under the assumptions (i)-(iii), the fractional integral equation (3.7) has at least one weak solution $x \in C[I, E]$. 
4. Fractional differential equations

In this section we establish existence results for the fractional differential equations (1.1)-(1.2) and (1.3)-(1.4) in the reflexive Banach space $E$.

4.1. Weak solution

Consider the integral equation

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \, ds, \quad t \in I,$$

where the integral is in the sense of weakly Riemann.

**Lemma 4.1.** Let $\alpha \in (0,1)$. A function $x$ is a weak solution of the fractional integral equation (4.9) if and only if $x$ is a solution of the problem (1.1)-(1.2).

**Proof.** Integrating (1.1)-(1.2) we obtain the integral equation (4.9). Operating by $R^{D^\alpha}$ on (4.9) we obtain the problem (1.1)-(1.2). So the equivalent between (1.1)-(1.2) and the integral equation (4.9) is proved and then the results follows from Theorem 3.3. □

4.2. Mild solution

Consider now the problem (1.3)-(1.4). According to Definitions 1.1 and 1.3, it is suitable to rewrite the problem (1.3)-(1.4) in the integral equation

$$x(t) = x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \, ds, \quad t \in I.$$

**Definition 4.2.** By the mild solution of the problem (1.3)-(1.4), we mean that the function $x \in C[I, E]$ which satisfies the corresponding integral equation of (1.3)-(1.4) which is (4.10).

**Theorem 4.3.** If (i)-(iii) are satisfied, then the problem (1.3)-(1.4) has at least one mild solution $x \in C[I, E]$.

It is often the case that the problem (1.3)-(1.4) does not have a differentiable solution yet does have a solution, in a mild sense.

References


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