Journal of
Mathematics
and Applications
No 35, pp 53-67 (2012)

Close-to-convexity properties of basic hypergeometric functions using their Taylor coefficients

K. Raghavendar and *A. Swaminathan*[1]

Submitted by: *Jan Stankiewicz*

ABSTRACT: In this paper, we find the conditions on parameters a, b, c and q such that the basic hypergeometric function $z\phi(a,b;c;q,z)$ and its q-Alexander transform are close-to-convex (and hence univalent) in the unit disc $\mathbb{D} := \{z : \ |z| < 1\}$.

AMS Subject Classification: *30C45, 33C05, 33D05, 33D15*
Key Words and Phrases: *Basic hypergeometric functions, Gaussian hypergeometric functions, close-to-convex functions, starlike functions, convex functions*

1 Introduction and Notation

Most of the mathematical functions which are encountered in numerous contexts are of hypergeometric type. The ordinary or Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by the series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} z^n, \quad |z| < 1, $$

where a, b, c are complex numbers such that $c \neq 0, -1, -2, -3, ...$, $(a,0) = 1$ for $a \neq 0$ and

$$ (a, n+1) = (a+n)(a,n), \quad n = 0, 1, 2, \cdots . $$

In the exceptional case $c = -p$, $p = 0, 1, 2, \cdots$, $F(a,b;c;z)$ is defined if $a = -m$ or $b = -m$, where $m = 0, 1, 2, \cdots$ and $m \leq p$. Heine (see [1, 5]) defined 'q-analogue' or 'basic analogue' of Gaussian hypergeometric function in the following way

$$ {}_2\Phi_1(a,b;c;q;z) = 1 + \frac{(1-q^a)(1-q^b)}{(1-q^c)(1-q)} z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q^c)(1-q^{c+1})(1-q)(1-q^2)} z^2 + \cdots , $$

[1] Corresponding author

where $|q| < 1$. For the base q, $0 < q < 1$, define

$$(a;q) = \frac{1-q^a}{1-q}, \quad (0;q) = 1.$$

Clearly, by L'-Hospitals' rule, we have

$$(a;q) \to a \text{ as } q \to 1,$$

and the basic factorial notation is

$$(a;q)! = (1;q)(2;q)\cdots(n;q).$$

Now we write

$$\begin{aligned}(1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1}) &= (a;q)(a+1;q)\cdots(a+n-1;q)(1-q)^n \\ &= (a;q)_n(1-q)^n, \text{ say,}\end{aligned}$$

and thus in the limiting case $q \to 1$, we have

$$\lim_{q\to 1}(a;q)_n = \lim_{q\to 1}\prod_{j=1}^{n}\frac{1-q^{a+j-1}}{(1-q)^n} = (a,n).$$

With this observation, the Heine's series or the q-analogue of Gauss function defined above takes the following form [17]:

$${}_2\Phi_1(a,b;c;q;z) = \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}z^n, \quad |z| < 1.$$

We remark that in the limiting case $q \to 1$, the function ${}_2\Phi_1(a,b;c;q;z)$ reduces to ${}_2F_1(a,b;c;z)$.

The geometric properties of ${}_2F_1(a,b;c;z)$ for various values of a, b and c are well known. For details, we refer to [7, 9, 10, 13, 14] and references therein. Similar study about ${}_2\phi_1(a,b;c;q;z)$ is not available in the literature, except [6, 15, 16]. Hence the main objective of this work is to find the geometric properties of ${}_2\phi_1(a,b;c;q;z)$ from the parameters a, b and c for $0 < q < 1$. For this purpose the q-Gamma function $\Gamma_q(x)$ [2], which is the q-generalization of the Gamma function and defined by

$$\Gamma_q(x)_: = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad 0 < q < 1,$$

is used.

Throughout the sequel, we always asuume that $z \in \mathbb{D}$ where $\mathbb{D}$ is the unit disc given by $\{z : |z| < 1\}$. The class of normalized analytic functions

$$\mathcal{A} = \left\{ f : \mathbb{D} \to \mathbb{C} \,|\, f(z) = z + \sum_{n=2}^{\infty} A_n z^n \right\} \tag{1.1}$$

has been studied extensively, together with its subclass of univalent (schlicht) functions

$$\mathcal{S} = \{f \in \mathcal{A} \,|\, f \text{ is one-to-one in } \mathbb{D}\}. \tag{1.2}$$

For $f \in \mathcal{A}$, the q-difference operator of the basic differentiation is given by

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{z(1-q)}, & z \neq 0 \\ f'(0), & z = 0. \end{cases}$$

Clearly $D_q \to \dfrac{d}{dq}$ as $q \to 1$. A function $f \in \mathcal{A}$ is called starlike ($f \in \mathcal{S}^*$) if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \tag{1.3}$$

and $f \in \mathcal{A}$ is called close-to-convex ($f \in \mathcal{K}$) if there exists $g \in \mathcal{S}^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{1.4}$$

Using q-difference operator the authors in [6] generalize the family $\mathcal{S}^*$ as follows:

Definition 1.1. *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{P}S_q$ if*

$$\left|\frac{z(D_q f)(z)}{f(z)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}. \tag{1.5}$$

Clearly $\mathcal{P}S_q$ reduces to $\mathcal{S}^*$ as $q \to 1^-$. Not much is known about the class $\mathcal{P}S_q$, except what is discussed in [6] for the inclusion of the functions ${}_2\Phi_1(a, b; c; q; z)$ and for the study of certain continued fraction expansions given by [11]. Recently the second author, among other results, studied [15, 16] certain continued fraction expansion for ${}_2\Phi_1(a, b; c; q; z)$ and used it to improve the results given in [6]. We now generalize the class $\mathcal{K}$ in the spirit as the Definition 1.1 generalizes $\mathcal{S}^*$.

Definition 1.2. *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{P}K_q$ if there exists $g \in \mathcal{S}^*$ such that*

$$\left|\frac{z(D_q f)(z)}{g(z)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}. \tag{1.6}$$

We observe that (1.6) reduces to (1.4) as $q \to 1^-$ and hence in the limiting case $\mathcal{P}K_q$ reduces to $\mathcal{K}$. Particular choice of the function g used in the study of $\mathcal{P}K_q$ are interesting. According to Frideman [4], there exists only nine functions of the class $\mathcal{S}$ whose coefficients are rational integers. They are

$$\frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2} \tag{1.7}$$

together with the identity function. It is easy to see that each of these functions maps the disc $\mathbb{D}$ onto a starlike domain and the last fact is easy to see from the analytic characterization given by (1.3). We remark that each of these functions plays an important role in function theory since these together with its rotation are extremal for interesting subfamilies of $\mathcal{S}$. We first state few useful criteria for a normalized power series ($A_0 = 0,\ A_1 = 1,$) defined by

$$f(z) = \sum_{n=0}^{\infty} A_n z^n, \quad B_n = \frac{(1-q^n)A_n}{1-q}, \tag{1.8}$$

to belong to $\mathcal{P}K_q$.

Lemma 1.1. *Let f be defined by* (1.8) *and $B_n = \dfrac{(1-q^n)A_n}{1-q}$. Then we have the following:*

(1) $\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq 1$ *implies $f \in \mathcal{P}K_q$ with $g(z) = z/(1-z)$.*

(2) $\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| \leq 1$ *implies $f \in \mathcal{P}K_q$ with $g(z) = z/(1-z+z^2)$.*

(3) $\sum_{n=1}^{\infty} |B_{n-1} - 2B_n + B_{n+1}| \leq 1$ *implies $f \in \mathcal{P}K_q$ with $g(z) = z/(1-z)^2$.*

(4) $\sum_{n=1}^{\infty} |B_{n-1} - B_{n+1}| \leq 1$ *implies $f \in \mathcal{P}K_q$ with $g(z) = z/(1-z^2)$.*

Proof. (1) Suppose that $\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq 1$.

The power series converges for $|z| < 1$. Since

$$\begin{aligned} |B_n| = \left| \sum_{k=1}^{n-1} (B_k - B_{k+1}) - 1 \right| &\leq \sum_{k=1}^{n-1} |B_k - B_{k+1}| + 1 \\ &\leq \sum_{k=1}^{\infty} |B_k - B_{k+1}| + 1 \leq 2 \end{aligned}$$

and

$$|B_n| = \left| \frac{(1-q^n)A_n}{1-q} \right| \leq 2, \tag{1.9}$$

we have

$$|A_n| \leq \frac{2}{1 + q + \cdots + q^{n-1}}.$$

Thus, by applying Root test, the radius of convergence of $f(z)$ is seen to be unity and so f is analytic in $\mathbb{D}$. Next we show that f belongs to $\mathcal{P}K_q$ with respect to the starlike function $g(z) = z/(1-z)$. For this we need to show that f satisfies the condition

$$\left|(1-z)(D_qf)(z) - \frac{1}{1-q}\right| \le \frac{1}{1-q}, \quad z \in \mathbb{D}. \tag{1.10}$$

By (1.8) and the defintion of q-difference operator, the above inequality can be rewritten in the equivalent form

$$T_q := \frac{1}{1-q} - \left|1 + \sum_{n=1}^{\infty}(B_{n+1} - B_n)z^n - \frac{1}{1-q}\right| \ge 0.$$

Applying triangle inequality, we find that

$$\begin{aligned} T_q &\ge \frac{1}{1-q} - \left|1 - \frac{1}{1-q}\right| - \left|\sum_{n=1}^{\infty}(B_{n+1} - B_n)z^n\right| \\ &\ge \frac{1}{1-q} - \frac{q}{1-q} - \sum_{n=1}^{\infty}|B_{n+1} - B_n||z|^n \\ &\ge 1 - \sum_{n=1}^{\infty}|B_{n+1} - B_n| \ge 0, \qquad \text{by hypothesis.} \end{aligned}$$

Thus (1.10) holds for all $z \in \mathbb{D}$. Hence $f \in \mathcal{P}K_q$ with $g(z) = z/(1-z)$.

Next we prove (2) and the rest follows similarly. Assume that

$$\sum_{n=1}^{\infty}|B_{n-1} - B_n + B_{n+1}| \le 1.$$

This implies that the power series converges for $|z| < 1$. Since

$$\begin{aligned} |B_n| &= \left|\sum_{k=2}^{n-2} B_k - \sum_{k=1}^{n-1}[B_{k-1} - B_k + (k+1)B_{k+1}]\right| \\ &\le \sum_{k=2}^{n-2}|B_k| + \sum_{k=1}^{n-1}|B_{k-1} - B_k + B_{k+1}| \le 1 + \sum_{k=2}^{\infty}|B_k| \end{aligned}$$

which is less than or equal to a finite quantity and hence the radius of convergence, by Root test, is unity. Taking $g(z) = z/(1-z+z^2)$, to prove $f \in \mathcal{P}K_q$ with respect to $g(z)$, we need to show that f satisfies the condition

$$\left|(1-z+z^2)(D_qf)(z) - \frac{1}{1-q}\right| \le \frac{1}{1-q}, \qquad z \in \mathbb{D},$$

which after some computation is seen to be equivalent to

$$S_q := \frac{1}{1-q} - \left|1 - \sum_{n=1}^{\infty}[-B_{n-1} + nB_n - B_{n+1}]z^n - \frac{1}{1-q}\right| \ge 0.$$

As in the first case, we use triangle inequality and obtain that for $|z| < 1$, $S_q \geq 0$. Hence $f \in \mathcal{P}K_q$ with $g(z) = z/(1 - z + z^2)$. This completes the proof. $\square$

The following lemma is immediate from the proof of Lemma 1.1.

Lemma 1.2. *Let B_n be as in Lemma 1.1 and*

$$f(z) = \sum_{n=0}^{\infty} (-1)^{n-1} A_n z^n \quad (A_0 = 0,\ A_1 = 1). \tag{1.11}$$

Then we have the following:

(1) $\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq 1$ *implies* $f \in \mathcal{P}K_q$ *with* $g(z) = z/(1 + z)$.

(2) $\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| \leq 1$ *implies* $f \in \mathcal{P}K_q$ *with* $g(z) = z/(1 + z + z^2)$.

(3) $\sum_{n=1}^{\infty} |B_{n-1} - 2B_n + B_{n+1}| \leq 1$ *implies* $f \in \mathcal{P}K_q$ *with* $g(z) = z/(1 + z)^2$.

(4) $\sum_{n=1}^{\infty} |B_{n-1} - B_{n+1}| \leq 1$ *implies* $f \in \mathcal{P}K_q$ *with* $g(z) = z/(1 + z^2)$.

Note that as $q \to 1$, Lemma 1.1 and Lemma 1.2 give criteria for close-to-convexity with reference to the eight different starlike functions defined by (1.7). In the special case when $q \to 1$, Lemma 1.1 gives results of Ozaki [12] (see also [8]) and for which applications have been obtained related to the univalency question of the Gaussian and the confluent hypergeometric functions by various authors. For example, we refer to [13, 14] and references therein.

Theorem 1.1. *If a and b are related by any one of the following conditions*

1. *(a)* $(1 - q^a)(1 - q^b) > (1 - q)$,

  (b) $\dfrac{\Gamma_q(a + b)}{\Gamma_q(a)\Gamma_q(b)} \leq \dfrac{2}{q}$.

2. *(a)* $(1 - q^{a-1})(1 - q^{b-1}) < -(1 - q)$ *and* $a + b > 2$,

  (b) $\dfrac{\Gamma_q(a + b)}{\Gamma_q(a)\Gamma_q(b)} \geq 0$.

Then the function $z\phi(a, b; a + b; q, z)$ belongs to $\mathcal{P}K_q$ with respect to $\dfrac{z}{1 - z}$.

Proof. Consider $\phi(a,b;a+b;q,z)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(a+b;q)_n(q;q)_n}z^n$, then

$$\begin{aligned} z\phi(a,b;a+b;q,z) &= z+\sum_{n=2}^{\infty}A_n z^n \\ &= z+\sum_{n=2}^{\infty}\frac{(a;q)_{n-1}(b;q)_{n-1}}{(a+b;q)_{n-1}(q;q)_{n-1}}z^n, \end{aligned} \tag{1.12}$$

therefore $B_n=\frac{1-q^n}{1-q}\frac{(a;q)_{n-1}(b;q)_{n-1}}{(a+b;q)_{n-1}(q;q)_{n-1}}$. We further write

$$B_{n+1}-B_n=\frac{1}{1-q}\frac{(a;q)_{n-1}(b;q)_{n-1}}{(a+b;q)_n(q;q)_n}f(q,n),$$

where

$$f(q,n)=(1-q^{n+1})(1-q^{a+n-1})(1-q^{b+n-1})-(1-q^n)^2(1-q^{a+b+n-1}).$$

If we take $S:=\sum_{n\geq 1}\left|B_{n+1}-B_n\right|$, then from Lemma (1.1), it is sufficient to show that $S\leq 1$.

We assume that the first hypothesis of the theorem is true. Now writing

$$\begin{aligned} f(q,n) &> (1-q^n)(1-q^{a+n-1})(1-q^{b+n-1})-(1-q^n)^2(1-q^{a+b+n-1}) \\ &= (1-q^n)\Big(((1-q^{a+n-1})(1-q^{b+n-1})-(1-q^n)(1-q^{a+b+n-1})\Big), \end{aligned}$$

to show that $f(q,n)>0$, it is enough to show that

$$(1-q^{a+n-1})(1-q^{b+n-1})-(1-q^n)(1-q^{a+b+n-1})>0.$$

Rewriting

$$\begin{aligned} &(1-q^{a+n-1})(1-q^{b+n-1})-(1-q^n)(1-q^{a+b+n-1}) \\ =&q^{n-1}((1-q^a)(1-q^b)-(1-q))+q^{a+b+2n-2}(1-q) \end{aligned}$$

we can easily see that the first term is positive from the given hypothesis $1(a)$, and the term $q^{a+b+2n-2}(1-q)$, which is positive for $n\geq 1$, hence $f(q,n)$ is positive. Now,

$$\begin{aligned} S &= \frac{1}{1-q}\sum_{n=1}^{\infty}\frac{(a;q)_{n-1}(b;q)_{n-1}}{(a+b;q)_n(q;q)_n}f(q,n) \\ &= -1+\frac{q(1-q^a)(1-q^b)}{1-q}\sum_{n=1}^{\infty}\frac{(a;q)_{n-1}(b;q)_{n-1}}{(a+b;q)_n(q;q)_n} \\ &= -1+q\frac{\Gamma_q(a+b)}{\Gamma_q(a)\Gamma_q(b)}\leq 1, \end{aligned}$$

from the hypothesis $1(b)$. Now considering the second hypothesis and observing that $f(q,n)$ is q^n multiple of

$$2-q-q^{a-1}-q^{b-1}+q^{a+b-1} \\ -q^n((1-q^a)(1-q^b)-q^{a+b}-q^{a+b-2}+2q^{a+b-1})$$

and considering

$$\begin{aligned} & q^n((1-q^a)(1-q^b)-q^{a+b}-q^{a+b-2}+2q^{a+b-1}) \\ = \;& q^n(1-q^a-q^b-q^{a+b-2}+2q^{a+b-1}) \\ = \;& q^n(1-q^a)(1-q^b)-q^{a+b-1+n}(1-q)+q^{a+b+n-2}(1-q)) \end{aligned}$$

we see that the first two terms of the above expression are positive for all $a,b>0$ and $n\geq 1$. Hence to show that $f(q,n)$ is negative it is enough to show that

$$2-q-q^{a-1}-q^{b-1}+q^{a+b-1}+q^{a+b+n-2}(1-q)<0$$

which is clearly true from hypothesis $2(a)$ by taking $a+b>2$ and using the inequality $q^{a+b+n-2}<q^{a+b-2}$. Now it is easy to see that

$$S=\frac{1}{1-q}\sum_{n=1}^{\infty}\frac{(a;q)_{n-1}(b;q)_{n-1}}{(a+b;q)_n(q;q)_n}f(q,n)=1-q\frac{\Gamma_q(a+b)}{\Gamma_q(a)\Gamma_q(b)}\leq 1$$

is true from hypothesis $2(b)$ and the proof is complete. □

The following corollary is immediate.

Corollary 1.1. *Let* $f(z)=z+\sum_{n=2}^{\infty}b_nz^n$ *and*

$$2\geq B_2\geq \ldots B_n-(n-2)\geq B_{n+1}-(n-1)\ldots\geq 0$$

or

$$0\leq B_2\leq B_3+1\leq B_4+2\ldots B_n+(n-2)\leq B_{n+1}+(n-1)\leq 2.$$

then $f\in\mathcal{P}K_q$ *with* $g(z)=z/(1-z)$.

Before proceeding for the next result, we give a list of functions.

$$\begin{aligned} g_1(q,a,b) &= ((1+q^a)(1+q^b)-(1+q^b))q-(1+q), \\ g_2(q,a,b) &= (1-q^{a-1})(1-q^{b-1}) \qquad \text{and} \\ g_3(q,a,b) &= g_1(q,a,b)(1-q)-1+q)(1-q^a)(1-q^b)-g_2(q,a,b)(1-q)^2. \end{aligned}$$

Theorem 1.2. *If a and b are related by any one of the following conditions*

1. (a) $(1-q^{a-1})(1-q^{b-1})<-(1-q)(1-q^{a+b-2})$

(b) $(1-q^a)(1-q^b) > (1-q)^2 q^{a+b-2}$ *and*

(c) $q\dfrac{(1-q^a)(1-q^b)\Gamma_q(a+b)}{\Gamma_q(a+2)\Gamma_q(b+2)(1-q)^4}\, g_3(q,a,b) < 0.$

2. *(a)* $(1-q^{a-1})(1-q^{b-1}) > -(1-q)(1-q^{a+b-2})$

(b) $(1-q^a)(1-q^b) < (1-q)^2 q^{a+b-2}$, *and*

(c) $q\dfrac{(1-q^a)(1-q^b)\Gamma_q(a+b)}{\Gamma_q(a+2)\Gamma_q(b+2)(1-q)^4}\, g_3(q,a,b) > -2.$

Then the function $z\phi(a,b;a+b;q,z)$ *belongs to* $\mathcal{P}K_q$ *with respect to* $\dfrac{z}{1-z^2}$.

Proof. Consider $\phi(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n$, then

$$z\phi(a,b;c;q,z) = z + \sum_{n=2}^{\infty} A_n z^n z + \sum_{n=2}^{\infty} \frac{(a;q)_{n-1}(b;q)_{n-1}}{(c;q)_{n-1}(q;q)_{n-1}} z^n,$$

therefore $B_n = \dfrac{1-q^n}{1-q}\dfrac{(a;q)_{n-1}(b;q)_{n-1}}{(c;q)_{n-1}(q;q)_{n-1}}$, which gives

$$B_{n-1} - B_{n+1} = \frac{1}{1-q}\frac{(a;q)_{n-2}(b;q)_{n-2}}{(c;q)_n (q;q)_n} g(q,n),$$

where

$$\begin{aligned} g(q,n) =& (1-q^n)(1-q^{n-1})^2(1-q^{a+b+n-2})(1-q^{a+b+n-1}) \\ &- (1-q^{n+1})(1-q^{a+n-1})(1-q^{a+n-2})(1-q^{b+n-1})(1-q^{b+n-2}). \end{aligned}$$

Now we take

$$S := \sum_{n\geq 1} \left|B_{n-1} - B_{n+1}\right|,$$

from Lemma 1.1 it is sufficient to show that $S \leq 1$.

For the first part, writing

$$B_{n-1} - B_{n+1} = (B_{n-1} - B_n) + (B_n - B_{n+1}),$$

we have

$$B_{n-1} - B_n = \frac{(a;q)_{n-2}(b;q)_{n-2}}{(1-q)(a+b;q)_{n-2}(q;q)_{n-2}} M(f,q,n)$$

where

$$M(f,q,n) = (1-q^{n-1})(1-q^{a+b+n-2})(1-q^{n-1}) - (1-q^n)(1-q^{a+n-2})(1-q^{b+n-2})$$

$$= - q^{n-1}(A) + q^{2n-2}(B)$$

with

$$A =(1-q)(1-q^{a+b-2}) + (1-q^{a-1})(1-q^{b-1}) \tag{1.13}$$

and

$$B =(1-q^a)(1-q^b) - (1-q)^2 q^{a+b-2}. \tag{1.14}$$

This gives $B_{n-1} - B_n$ positive, since $A < 0$ from $1(a)$ and $B > 0$ from $1(b)$ of the hypotheses of the theorem. Similarly

$$B_n - B_{n+1} = \frac{(a;q)_{n-2}(b;q)_{n-2}}{(1-q)(a+b;q)_{n-2}(q;q)_{n-2}} M_1(f,q,n),$$

where

$$\begin{aligned} M_1(f,q,n) =&(1-q^n)(1-q^{a+b+n-1})(1-q^n) - (1-q^{n+1})(1-q^{a+n-1})(1-q^{b+n-1}) \\ =& - q^n(A) + q^{2n}(B). \end{aligned}$$

Hence

$$\begin{aligned} B_{n-1} - B_{n+1} =&(B_{n-1} - B_n) + (B_n + B_{n+1}) \\ &= -(1+q)q^{n-1}A + (1+q^2)q^{2n-2}B = g(q,n) \end{aligned}$$

which is positive from the hypothesis, since $A < 0$ from $1(a)$, $B > 0$ from $1(b)$. Combining these we have $B_{n-1} - B_{n+1}$ is positive. Further

$$\begin{aligned} S =& \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(c;q)_n(q;q)_n} g(q,n) \\ =&1 + \frac{q(1-q^a)(1-q^b)}{1-q} \left(g_1(q,a,b) \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n(q;q)_{n-2}} q^{2n-4} \right. \\ &- (1+q) \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n(q;q)_{n-3}} q^{n-2} \\ &\left. - g_2(q,a,b) \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n(q;q)_{n-1}} q^{3n-3} \right) \\ =&1 + q \frac{(1-q^a)(1-q^b)\Gamma_q(a+b)}{\Gamma_q(a+2)\Gamma_q(b+2)(1-q)^4} g_3(q,a,b) \leq 1, \end{aligned}$$

from $1(c)$ of the hypothesis.

Proceeding similar to the first part, we can easily see that $g(q,n)$ is negative from the hypothesis $2(a)$ and $2(b)$. Hence

$$S = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(c;q)_n(q;q)_n} g(q,n)$$

$$= -1 - q\frac{(1-q^a)(1-q^b)\Gamma_q(a+b)}{\Gamma_q(a+2)\Gamma_q(b+2)(1-q)^4} g_f(q,n) \le 1$$

from $2(c)$ of the hypothesis and the proof is complete. □

Theorem 1.3. *The function $z\phi(a,b;a+b;q,z)$ belongs to $\mathcal{P}K_q$ with respect to the starlike function $\frac{z}{(1-z)^2}$, whenever*

$$\Big((1-q^{a-1})(1-q^{b-1}) + (1-q)(1-q^{a+b-2})\Big) \times \Big((1-q^a)(1-q^b) - (1-q)^2 q^{a+b-2}\Big) > 0. \tag{1.15}$$

Proof. Consider $\phi(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} z^n$, then

$$z\phi(a,b;c;q,z) = z + \sum_{n=2}^{\infty} A_n z^n z + \sum_{n=2}^{\infty} \frac{(a;q)_{n-1}(b;q)_{n-1}}{(c;q)_{n-1}(q;q)_{n-1}} z^n,$$

therefore $B_n = \frac{1-q^n}{1-q}\frac{(a;q)_{n-1}(b;q)_{n-1}}{(c;q)_{n-1}(q;q)_{n-1}}$. Let $S := \sum_{n\ge 1} \left|B_{n-1} - 2B_n + B_{n+1}\right|$. Then, from Lemma 1.1, it is sufficient to show that $S \le 1$. Infact, we show that $|S| = 1$.

Now

$$B_{n-1} - 2B_n + B_{n+1} = \frac{1}{1-q}\frac{(a;q)_{n-2}(b;q)_{n-2}}{(c;q)_n(q;q)_n} h(q,n)$$

where

$$\begin{aligned} h(q,n) &= (1-q^{n-1})(1-q^{a+b+n-1})(1-q^{a+b+n-2})(1-q^n)(1-q^{n-1}) \\ &\quad - 2(1-q^n)(1-q^{a+n-2})(1-q^{b+n-2})(1-q^{a+b+n-1})(1-q^n) \\ &\quad + (1-q^{n+1})(1-q^{a+n-1})(1-q^{a+n-2})(1-q^{b+n-1})(1-q^{b+n-2}), \\ &= (1-q)(q^{n-1}A + (1+q)q^{2n-2}B), \end{aligned}$$

where A and B are respectively, as in (1.13) and (1.14). Thus, taking

$$\begin{aligned} (1-q^{a-1})(1-q^{b-1}) &> -(1-q)(1-q^{a+b-2}) \qquad \text{and} \\ (1-q^a)(1-q^b) &> (1-q)^2 q^{a+b-2} \end{aligned}$$

satisfies the hypothesis (1.15) of the theorem, which means we get $B_{n-1} - 2B_n + B_{n+1}$ is positive. On the other hand, if we take

$$\begin{aligned} (1-q^{a-1})(1-q^{b-1}) &< -(1-q)(1-q^{a+b-2}) \qquad \text{and} \\ (1-q^a)(1-q^b) &< (1-q)^2 q^{a+b-2} \end{aligned}$$

we again see that the hypothesis (1.15) of the theorem is satisfied to yield $B_{n-1} - 2B_n + B_{n+1}$ negative. This means $B_{n-1} - 2B_n + B_{n+1} \neq 0$.

Now

$$\begin{aligned}
|S| &= \left| \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(c;q)_n (q;q)_n} h(q,n) \right| \\
&= \Bigg| -1 + q(1-q^a)(1-q^b) \left(\frac{(1-q^a)(1-q^b)}{1-q} \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n (q;q)_{n-1}} q^{2n-3} \right. \\
&\left. - \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n (q;q)_{n-1}} q^{n-2} + \sum_{n=1}^{\infty} \frac{(a;q)_{n-2}(b;q)_{n-2}}{(a+b;q)_n (q;q)_{n-1}} q^{a+b+3n-5} \right) \Bigg| \\
&= 1,
\end{aligned}$$

which satisfies Lemma 1.1 and the proof is complete. □

We define the q-Alexander transform, analogous to the Alexander transform [3] in the following way. Given $f \in \mathcal{A}$, the q-Alexander transform is given by

$$\Lambda_{f,q}(z) = \int_0^z \frac{f(t)}{t} d_q(t), \qquad f \in \mathcal{A}, \quad z \in \mathbb{D}. \tag{1.16}$$

Hence, for $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$, we see that

$$\Lambda_{f,q}(z) = z + \sum_{n=2}^{\infty} A_n \frac{1-q}{1-q^n} z^n.$$

With this, we give our next result.

Theorem 1.4. *Let a, b and c satisfy any one of the following properties.*

1. $a, b \in (1, \infty)$ *and* $\Gamma_q(a+b-1) \leq 2\Gamma_q(a)\Gamma_q(b)$,
2. $a \in (0,1)$, $b \in (1-a, 1)$ *and* $\Gamma_q(a+b-1) \leq 2\Gamma_q(a)\Gamma_q(b)$, *and*
3. $a \in (0,1)$, $b \in (1, \infty)$ *and* $\dfrac{\Gamma_q(a+b-1)}{\Gamma_q(a)\Gamma_q(b)} \geq 0$.

Then the q-Alexander transform (1.16) *of the function $z\phi(a,b;a+b-1;q,z)$ is in PK_q with $g(z) = \dfrac{z}{1-z}$.*

Proof. Given $f(z) = z\phi(a,b;a+b-1;q;z)$, $\Lambda_{f,q}(z)$ is given by

$$z + \sum_{n=2}^{\infty} \frac{(a;q)_n (b;q)_n}{(a+b-1;q)_n (q;q)_n} \frac{1-q}{1-q^n} z^n, \qquad z \in \mathbb{D}.$$

Then, in both the cases, viz., $a, b \in (1, \infty)$ and $a \in (0,1)$, $b \in (1-a, 1)$,

$$\big|B_{n+1} - B_n\big| = \Bigg| \frac{(a;q)_{n-1}(b;q)_{n-1}}{(a+b-1;q)_n(q;q)_n} \times \Big[(1-q^{a+n-1})(1-q^{b+n-1}) - (1-q^{a+b+n-2})(1-q^n)\Big]\Bigg|,$$

so that

$$\begin{aligned} S := \sum_{n=1}^{\infty} \big|B_{n+1} - B_n\big| &= \sum_{n=1}^{\infty} \frac{(a-1;q)_n(b-1;q)_n}{(a+b-1;q)_n(q;q)_n} q^n \\ &= \frac{\Gamma_q(a+b-1)}{\Gamma_q(a)\Gamma_q(b)} - 1 \leq 1, \end{aligned}$$

since $\Gamma_q(a+b-1) \leq 2\Gamma_q(a)\Gamma_q(b)$. In the case $a \in (0,1)$ and $b \in (1, \infty)$,

$$\begin{aligned} S := \sum_{n=1}^{\infty} \big|B_{n+1} - B_n\big| &= -\sum_{n=1}^{\infty} \frac{(a-1;q)_n(b-1;q)_n}{(a+b-1;q)_n(q;q)_n} q^n \\ &= 1 - \frac{\Gamma_q(a+b-1)}{\Gamma_q(a)\Gamma_q(b)} \leq 1, \end{aligned}$$

using $\dfrac{\Gamma_q(a+b-1)}{\Gamma_q(a)\Gamma_q(b)} \geq 0$ and the proof is complete. □

Corollary 1.2. *Let a and b satisfy any one of the following conditions*

1. $a, b \in (0, \infty)$ *and* $\Gamma_q(a+b-1) \leq 2\Gamma_q(a)\Gamma_q(b)$,
2. $a \in (-1, 0)$, $b \in (-1-a, 0)$ *and* $\Gamma_q(a+b-1) \leq 2\Gamma_q(a)\Gamma_q(b)$,
3. $a \in (-1, 0)$, $b \in (0, \infty)$ *and* $\dfrac{\Gamma_q(a+b-1)}{\Gamma_q(a)\Gamma_q(b)} \geq 0$.

Then the function $\dfrac{(1-q^{a+b})(1-q)}{(1-q^a)(1-q^b)}\big[z\phi(a,b;a+b;q,z)\big]$ *belongs to* PK_q *with respect to* $g(z) = \dfrac{z}{1-z}$.

Proof. The q-Alexander transform of $g(z) = z\phi(a+1, b+1; c+1; q, z)$ is

$$\Lambda_{g,f}(z) = \int_0^z \frac{g(t)}{t} d_q t = \frac{(1-q)(1-q^c)}{(1-q^a)(1-q^b)} \big[\phi(a,b;c;q,z) - 1\big]$$

and the results follow from Theorem 1.4 by replacing $a = a+1$, $b = b+1$ and $c = a+b+1$. □

References

[1] W.N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, New York, 1935.

[2] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications, 35, Cambridge Univ. Press, Cambridge, 1990.

[3] P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983.

[4] B. Friedman, *Two theorems on schlicht functions*, Duke Math. J. **13** (1946), 171–177.

[5] E. Heine, *Untersuchungen über die Reihe* . . . , J. Reine Angew. Math. 34 (1847) 285–328.

[6] M. E. H. Ismail, E. Merkes and D. Styer, *A generalization of starlike functions*, Complex Variables Theory Appl. **14** (1990), no. 1-4, 77–84.

[7] Küstner R., *Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order* α, Comput. Methods Funct. Theory **2** (2002), no. 2, 597–610.

[8] T. H. MacGregor, *Univalent power series whose coefficients have monotonic properties*, Math. Z. **112** (1969), 222–228.

[9] S. R. Mondal and A. Swaminathan, *Coefficient conditions for univalency and starlikeness of analytic functions*, J. Math. Appl. **31** (2009), 77–90.

[10] Saiful R. Mondal and A. Swaminathan, *Geometric properties of generalized Polylogarithm*, Integral Transforms Spec. Funct. **21** (2010) 691–701.

[11] F. Rønning, *A Szegő quadrature formula arising from q-starlike functions, in Continued fractions and orthogonal functions (Loen, 1992)*, 345–352, Dekker, New York.

[12] S. Ozaki, *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku A **2** (1935), 167–188.

[13] A. Swaminathan, *Inclusion theorems of convolution operators associated with normalized hypergeometric functions*, J. Comput. Appl. Math. **197** (2006), no. 1, 15–28.

[14] A. Swaminathan, *Univalent polynomials and fractional order differences of their coefficients*, J. Math. Anal. Appl. **353** (2009), no. 1, 232–238.

[15] A. Swaminathan, *Continued fraction expansion for certain hypergeometric functions*, Proceedings of the International Conference on Mathematical Sciences, Center for Mathematical Sciences, Pala, Kerala, 1–22.

[16] A. Swaminathan, *Pick functions and Chain Sequences for hypergeometric type functions*, Communicated for publication.

[17] N. M. Temme, *Special functions*, Wiley, New York, 1996.

DOI: 10.7862/rf.2012.5

K. Raghavendar
email: raghavendar248@gmail.com
Department of Mathematics
Indian Institute of Technology
Roorkee 247 667, Uttarkhand, INDIA

A. Swaminathan
email: swamifma@iitr.ernet.in
Department of Mathematics
Indian Institute of Technology
Roorkee 247 667, Uttarkhand, INDIA

Received 11.10.2011