Journal of
Mathematics
and Applications

JMA No 38, pp 85-104 (2015)

Approximate controllability of the impulsive semilinear heat equation[1]

Hugo Leiva and Nelson Merentes

ABSTRACT: In this paper we apply Rothe's Fixed Point Theorem to prove the interior approximate controllability of the following semilinear impulsive Heat Equation

$$\begin{cases} z_t = \Delta z + 1_\omega u(t,x) + f(t,z,u(t,x)), & \text{in} \quad (0,\tau] \times \Omega, t \neq t_k \\ z = 0, & \text{on} \quad (0,\tau) \times \partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega, \\ z(t_k^+,x) = z(t_k^-,x) + I_k(t_k, z(t_k,x), u(t_k,x)), & x \in \Omega, \end{cases}$$

where $k = 1,2,\dots,p$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, $z_0 \in L_2(\Omega)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω,the distributed control u belongs to $C([0,\tau];L_2(\Omega))$ and $f, I_k \in C([0,\tau] \times \mathbb{R} \times \mathbb{R};\mathbb{R})$, $k = 1,2,3,\dots,p$, such that

$$|f(t,z,u)| \leq a_0|z|^{\alpha_0} + b_0|u|^{\beta_0} + c_0, \quad u \in \mathbb{R}, z \in \mathbb{R}.$$

$$|I_k(t,z,u)| \leq a_k|z|^{\alpha_k} + b_k|u|^{\beta_k} + c_k, \quad k = 1,2,3,\dots,p, \quad u \in \mathbb{R}, z \in \mathbb{R}.$$

with $\frac{1}{2} \leq \alpha_k < 1$, $\frac{1}{2} \leq \beta_k < 1$, $\quad k = 0,1,2,3,\dots,p$. Under this condition we prove the following statement: For all open nonempty subsets ω of Ω the system is approximately controllable on $[0,\tau]$. Moreover, we could exhibit a sequence of controls steering the nonlinear system from an initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$.

AMS Subject Classification: *primary: 93B05; secondary: 93C10.*
Keywords and Phrases: *impulsive semilinear heat equation, approximate controllability, Rothe's fixed point Theorem.*

[1]This work has been supported by CDCHT-ULA-C-1796-12-05-AA and BCV

1 Introduction

There are many practical examples of impulsive control systems, a chemical reactor system with the quantities of different chemicals serve as the states, a financial system with two state variables of the amount of money in a market and the saving rates of a central bank and the growth of a population diffusing throughout its habitat is often modeled by reaction-diffusion equation, for which much has been done under the assumption that the system parameters related to the population environment, either are constant or change continuously.However, one may easily visualize situations in nature where abrupt changes such as harvesting, disasters and instantaneous stoking may occur. This observation motivates us to study the approximate controllability of the following Semilinear Impulsive Heat Equation

$$\begin{cases} z_t = \Delta z + 1_\omega u(t,x) + f(t,z,u(t,x)), & \text{in} \quad (0,\tau] \times \Omega, t \neq t_k \\ z = 0, & \text{on} \quad (0,\tau) \times \partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega, \\ z(t_k^+,x) = z(t_k^-,x) + I_k(t_k, z(t_k,x), u(t_k,x)), & x \in \Omega, \end{cases} \tag{1.1}$$

where $k = 1,2,\dots,p$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, $z_0 \in L_2(\Omega)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω,the distributed control u belongs to $C([0,\tau]; L_2(\Omega))$ and $f, I_k \in C([0,\tau] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, $k = 1,2,3,\dots,p$, such that

$$|f(t,z,u)| \leq a_0|z|^{\alpha_0} + b_0|u|^{\beta_0} + c_0, \qquad u, z \in \mathbb{R}. \tag{1.2}$$

$$|I_k(t,z,u)| \leq a_k|z|^{\alpha_k} + b_k|u|^{\beta_k} + c_k, \quad k = 1,2,3,\dots,p, \qquad u, z \in \mathbb{R}. \tag{1.3}$$

$$\frac{1}{2} \leq \alpha_k < 1, \frac{1}{2} \leq \beta_k < 1, \quad k = 0,1,2,3,\dots,p, \tag{1.4}$$

and

$$z(t_k,x) = z(t_k^+,x) = \lim_{t \to t_k^+} z(t,x), \quad z(t_k^-,x) = \lim_{t \to t_k^-} z(t,x).$$

In almost all reference on impulsive differential equations the natural space to work in is the Banach space

$$\begin{aligned} &PC([0,\tau]; Z) \\ &= \{z : J = [0,\tau] \to Z : z \in C(J'; Z), \exists z(t_k^+, \cdot), z(t_k^-, \cdot) \quad \text{and} \quad z(t_k, \cdot) = z(t_k^+, \cdot)\}, \end{aligned}$$

where $Z = L_2(\Omega)$ and $J' = [0,\tau] \backslash \{t_1, t_2, \dots, t_p\}$, endowed with the norm

$$\|z\| = \sup_{t \in [0,\tau]} |z(t,\cdot)|_Z,$$

with

$$\|z\|_Z = \sqrt{\int_\Omega \|z(x)\|^2 dx}, \quad \forall z \in Z = L_2(\Omega).$$

Definition 1.1 *(***Approximate Controllability***) The system (1.1) is said to be approximately controllable on $[0, \tau]$ if for every z_0, $z_1 \in Z = U = L_2(\Omega)$, $\varepsilon > 0$ there exists $u \in C([0, \tau]; U)$ such that the solution $z(t)$ of (1.1) corresponding to u verifies:*

$$z(0) = z_0 \quad \textit{and} \quad \|z(\tau) - z_1\|_Z < \varepsilon, \quad (\textit{Fig.2}),$$

where

$$\|z(\tau) - z_1\|_Z = \left(\int_\Omega |z(\tau, x) - z_1(x)|^2 dx \right)^{\frac{1}{2}}.$$

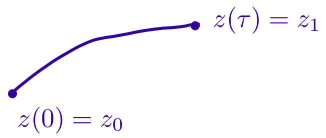

Fig.1

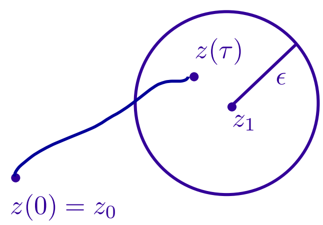

Fig.2

Definition 1.2 *(***Controllability to Trajectories***) The system (1.1) is said to be controllable to trajectories on $[0, \tau]$ if for every $z_0, \hat{z}_0 \in Z = U = L_2(\Omega)$ and $\hat{u} \in C([0, \tau]; U)$ there exists $u \in C([0, \tau]; U)$ such that the mild solution $z(t)$ of (1.1) corresponding to u verifies:*

$$z(\tau, z_0, u) = z(\tau, \hat{z}_0, \hat{u}), \quad (\textit{Fig.3}).$$

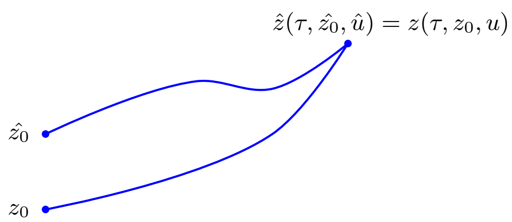

Fig.3

Definition 1.3 *(***Null Controllability***) The system (1.1) is said to be null controllable on $[0, \tau]$ if for every $z_0 \in Z = U = L_2(\Omega)$ there exists $C([0, \tau]; U)$ such that the mild solution $z(t)$ of (1.1) corresponding to u verifies:*

$$z(0) = z_0 \quad \textit{and} \quad z(\tau) = 0, \quad (\textit{Fig.4}).$$

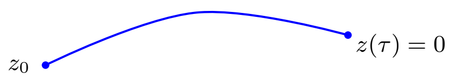

Fig.4

Remark 1.1 *It is clear that exact controllability of the system(1.1) implies approximate controllability, null controllability and controllability to trajectories of the system.But, it is well known ([2]) that due to the diffusion effect or the compactness of the semigroup generated by $-\Delta$, the heat equation can never be exactly controllable. We observe also that in the linear case controllability to trajectories and null controllability are equivalent. Nevertheless, the approximate controllability and the null controllability are in general independent. Therefore, in this paper we will be concentrated only on the study of the approximate controllability of the system(1.1).*

Recently the interior controllability of the semilinear heat equation (1.1) without impulses has been proved in [13], [14] and [15] under the following condition:

$$\sup_{(t,z,u)\in Q_\tau} |f(t,z,u) - az - cu| < \infty, \tag{1.5}$$

where $a, c \in I\!R$, with $c \neq -1$ and $Q_\tau = [0,\tau] \times I\!R \times I\!R$.
More recently, in [14], the approximate controllability of the semilinear heat equation (1.1) without impulses has been proved under the following non linear perturbation:

$$|f(t,z,u) - az| \leq c|u|^\beta + b, \quad \forall u, z \in I\!R, \quad |u|, |z| \geq R, \tag{1.6}$$

where $a, b, c \in I\!R$, $R > 0$ and $\frac{1}{2} \leq \beta < 1$. We note that, the interior approximate controllability of the linear heat equation

$$\begin{cases} z_t(t,x) = \Delta z(t,x) + 1_\omega u(t,x) & \text{in} \quad (0,\tau] \times \Omega, \\ z = 0, & \text{on} \quad (0,\tau) \times \partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega, \end{cases} \tag{1.7}$$

has been study by several authors, particularly by [22],[23],[24]; and in a general fashion in [12].
The controllability of Impulsive Evolution Equations has been studied recently for several authors, but most them study the exact controllability only, to mention: D.N. Chalishajar([4]), studied the exact controllability of impulsive partial neutral functional differential equations with infinite delay, B. Radhakrishnan and K. Balachandran([19]) studied the exact controllability of semilinear impulsive integrodifferential evolution systems with nonlocal conditions and S. Selvi, M. Mallika Arjunan([20]) studied the exact controllability for impulsive differential systems with finite delay. To our knowledge, there are a few works on approximate controllability of impulsive semilinear evolution equations, to mention: Lizhen Chen and Gang Li([5])

studied the Approximate controllability of impulsive differential equations with nonlocal conditions, using measure of noncompactness and Monch fixed point theorem, and assuming that the nonlinear term $f(t,z)$ does not depend on the control variable.

Finally, the approximate controllability of the system (1.1) follows from the approximate controllability of (1.7), the compactness of the semigroup generated by the Laplacian operator $-\Delta$, the conditions (1.2) and (1.5) satisfied by the nonlinear term f, I_k and the following results:

Proposition 1.1 *Let (X,Σ,μ) be a measure space with $\mu(X)<\infty$ and $1\le q<r<\infty$. Then $L_r(\mu)\subset L_q(\mu)$ and*

$$\|f\|_q \le \mu(X)^{\frac{r-q}{rq}}\|f\|_r, \quad f\in L_r(\mu). \tag{1.8}$$

Proof The proof of this proposition follows from Theorem I.V.6 from [3] by putting $p=\frac{r}{q}>1$ and considering the relation

$$\int_X (|f|^q)^p d\mu = \int_X |f|^r d\mu, \quad \forall f\in L_r(\mu).$$

$\square$

Theorem 1.1 *(Rothe's Fixed Theorem, [1],[9], [21]) Let E be a Banach space. Let $B\subset E$ be a closed convex subset such that the zero of E is contained in the interior of B. Let $\Phi : B\to E$ be a continuous mapping with $\Phi(B)$ relatively compact in E and $\Phi(\partial B)\subset B$. Then there is a point $x^*\in B$ such that $\Phi(x^*)=x^*$.*

The technique we use here to prove the approximate controllability of the linear part of equation (1.7) is based on the classical Unique Continuation for Elliptic Equations (see [18]) and the following lemma:

Lemma 1.1 *(see Lemma 3.14 from [6], pg. 62) Let $\{\alpha_j\}_{j\ge1}$ and $\{\beta_{i,j} : i = 1,2,\dots,m\}_{j\ge1}$ be two sequences of real numbers such that: $\alpha_1>\alpha_2>\alpha_3\cdots$. Then*

$$\sum_{j=1}^{\infty} e^{\alpha_j t}\beta_{i,j}=0, \quad \forall t\in[0,\tau], \quad i=1,2,\cdots,m$$

iff

$$\beta_{i,j}=0, \quad i=1,2,\cdots,m; j=1,2,\cdots,\infty.$$

2 Abstract Formulation of the Problem

In this section we choose a Hilbert space where system (1.1) can be written as an abstract differential equation; to this end, we consider the following results appearing in [6] pg.46, [8] pg.335 and [10] pg.147:

Let us consider the Hilbert space $Z=L_2(\Omega)$ and $0<\lambda_1<\lambda_2<...<\lambda_j\longrightarrow\infty$ the eigenvalues of $-\Delta$ with the Dirichlet homogeneous conditions, each one with finite

multiplicity γ_j equal to the dimension of the corresponding eigenspace. Then we have the following well known properties
(i) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of $A = -\Delta$.
(ii) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{2.1}$$

where $< \cdot, \cdot >$ is the inner product in Z and

$$E_j z = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}. \tag{2.2}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and $z = \sum_{j=1}^{\infty} E_j z, \quad z \in Z$.
(iii) $-A$ generates an analytic semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z \quad \text{and} \quad \|T(t)\| \leq e^{-\lambda_1 t}, \quad t \geq 0. \tag{2.3}$$

Consequently, system (1.1) can be written as an abstract impulsive differential equations in Z:

$$\begin{cases} z' = -Az + B_\omega u + f^e(t, z, u), & t \in (0, \tau], t \neq t_k, \quad z \in Z \\ z(0) = z_0, & \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k), u(t_k)), & k = 1, 2, 3, \ldots, p. \end{cases} \tag{2.4}$$

where $u \in C([0, \tau]; U)$, $U = Z$, $B_\omega : U \longrightarrow Z$, $B_\omega u = 1_\omega u$ is a bounded linear operator,$I_k^e, f^e : [0, \tau] \times Z \times U \to Z$, are defined by

$$I_k^e(t, z, u)(x) = I_k(t, z(x), u(x)), \quad f^e(t, z, u)(x) = f(t, z(x), u(x)), \ \forall x \in \Omega, k = 1, 2, \ldots, p.$$

On the other hand, from conditions (1.2) and (1.5) we get the following estimates.

Proposition 2.1 *Under the conditions (1.2)-(1.5) the functions $f^e, I_k^e : [0, \tau] \times Z \times U \to Z$,$k = 1, 2, 3, \ldots, p$, defined above satisfy $\forall u, z \in Z = L_2(\Omega)$:*

$$\|f^e(t, z, u)\|_Z \leq \tilde{a}_0 \|z\|_Z^{\alpha_0} + \tilde{b}_0 \|u\|_Z^{\beta_0} + \tilde{c}_0 \tag{2.5}$$

$$\|I_k^e(t, z, u)\|_Z \leq \tilde{a}_k \|z\|_Z^{\alpha_k} + \tilde{b}_k \|u\|_Z^{\beta_k} + \tilde{c}_k, \quad k = 1, 2, 3, \ldots, p. \tag{2.6}$$

Proof.

$$\begin{aligned} \|f^e(t, z, u)\|_Z^2 &= \int_\Omega |f(t, z(x), u(x))|^2 dx \\ &\leq \int_\Omega \left(a_0 |z(x)|^{\alpha_0} + b_0 |u(x)|^{\beta_0} + c_0\right)^2 dx \\ &\leq \int_\Omega (4a_0^2 |z(x)|^{2\alpha_0} + 4^2 b_0^2 |u(x)|^{2\beta_0} + 4^2 c_0^2) dx \\ &\leq 4a_0^2 \int_\Omega |z(x)|^{2\alpha_0} dx + 4^2 b_0^2 \int_\Omega |u(x)|^{2\beta_0} dx + 4^2 c_0^2 \mu(\Omega). \end{aligned}$$

Then

$$\begin{aligned}\|f^e(t,z,u)\|_Z &\leq 2a_0\left(\int_\Omega |z(x)|^{2\alpha_0}dx\right)^{\frac{1}{2}} + 4b_0\left(\int_\Omega |u(x)|^{2\beta_0}dx\right)^{\frac{1}{2}} + 4c_0\sqrt{\mu(\Omega)}\\ &= 2a_0\|z\|^{\alpha_0}_{L_{2\alpha_0}} + 4b_0\|z\|^{\beta_0}_{L_{2\beta_0}} + 4c_0\sqrt{\mu(\Omega)}\end{aligned}$$

Now, since $\frac{1}{2} \leq \alpha_0 < 1 \Leftrightarrow 1 \leq 2\alpha_0 < 2$ and $\frac{1}{2} \leq \beta_0 < 1 \Leftrightarrow 1 \leq 2\beta_0 < 2$ applying proposition 1.1, we obtain that:

$$\|f^e(t,z,u)\|_Z \leq 2a_0\mu(\Omega)^{\frac{1-\alpha_0}{\alpha_0}}\|z\|^{\alpha_0}_Z + 2b_0\mu(\Omega)^{\frac{1-\beta_0}{\beta_0}}\|u\|^{\beta_0}_Z + 4c_0\sqrt{\mu(\Omega)}.$$

Analogously, we obtain the following estimate for $k = 1,2,3,\ldots,p$

$$\|I^e_k(t,z,u)\|_Z \leq 2a_k\mu(\Omega)^{\frac{1-\alpha_k}{\alpha_k}}\|z\|^{\alpha_k}_Z + 2b_k\mu(\Omega)^{\frac{1-\beta_k}{\beta_k}}\|u\|^{\beta_k}_Z + 4c_k\sqrt{\mu(\Omega)},$$

which completes the proof. ∎

3 Controllability of the Linear Equation without Impulses

In this section we shall present some characterization of the interior approximate controllability of the linear heat equations without impulses. To this end, we note that, for all $z_0 \in Z$ and $u \in L_2(0,\tau;U)$ the initial value problem

$$\begin{cases} z' = -Az + B_\omega u(t), & z \in Z,\\ z(0) = z_0, \end{cases} \tag{3.1}$$

where the control function u belongs to $L_2(0,\tau;U)$, admits only one mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)B_\omega u(s)ds, \quad t \in [0,\tau]. \tag{3.2}$$

Definition 3.1 *For system (3.1) we define the following concept: The controllability map (for $\tau > 0$) $G : L_2(0,\tau;U) \longrightarrow Z$ is given by*

$$Gu = \int_0^\tau T(\tau-s)B_\omega u(s)ds. \tag{3.3}$$

whose adjoint operator $G^* : Z \longrightarrow L_2(0,\tau;Z)$ is given by

$$(G^*z)(s) = B^*_\omega T^*(\tau-s)z, \quad \forall s \in [0,\tau], \quad \forall z \in Z. \tag{3.4}$$

Therefore, the Grammian operator $\mathcal{W} : Z \to Z$ is given

$$\mathcal{W}z = GG^*z = \int_0^\tau T(\tau-s)B_\omega B^*_\omega T^*(\tau-s)ds. \tag{3.5}$$

The following lemma holds in general for a linear bounded operator $G : W \to Z$ between Hilbert spaces W and Z.

Lemma 3.1 *(see [6], [7] and [12]) The equation (3.1) is approximately controllable on* $[0,\tau]$ *if, and only if, one of the following statements holds:*

a) $\overline{\mathrm{Rang}(G)} = Z$.

b) $Ker(G^*) = \{0\}$.

c) $\langle GG^*z, z\rangle > 0$, $z \neq 0$ *in* Z.

d) $\lim_{\alpha\to 0^+} \alpha(\alpha I + GG^*)^{-1}z = 0$.

e) $B_\omega^* T^*(t)z = 0, \quad \forall t \in [0,\tau], \quad \Rightarrow z = 0$.

f) For all $z \in Z$ *we have* $Gu_\alpha = z - \alpha(\alpha I + GG^*)^{-1}z$, *where*

$$u_\alpha = G^*(\alpha I + GG^*)^{-1}z, \quad \alpha \in (0,1].$$

So, $\lim_{\alpha\to 0} Gu_\alpha = z$ *and the error* $E_\alpha z$ *of this approximation is given by*

$$E_\alpha z = \alpha(\alpha I + GG^*)^{-1}z, \quad \alpha \in (0,1].$$

Remark 3.1 *The Lemma 3.1 implies that the family of linear operators* $\Gamma_\alpha : Z \to L_2(0,\tau;U)$, *defined for* $0 < \alpha \leq 1$ *by*

$$\Gamma_\alpha z = B_\omega^* T^*(\cdot)(\alpha I + GG^*)^{-1}z = G^*(\alpha I + GG^*)^{-1}z, \tag{3.6}$$

is an approximate inverse for the right of the operator G *in the sense that*

$$\lim_{\alpha\to 0} G\Gamma_\alpha = I, \tag{3.7}$$

in the strong topology.

Proposition 3.1 *(See [15]) If* $\overline{\mathrm{Rang}(G)} = Z$, *then*

$$\sup_{\alpha>0} \|\alpha(\alpha I + GG^*)^{-1}\| \leq 1. \tag{3.8}$$

Remark 3.2 *The proof of the following theorem follows from foregoing characterization of dense range linear operators and the classical Unique Continuation for Elliptic Equations (see [18]), and it is similar to the one given in Theorem 4.1 in [14].*

Theorem 3.1 *System (3.1) is approximately controllable on* $[0,\tau]$. *Moreover, a sequence of controls steering the system (3.1) from initial state* z_0 *to an* ϵ *neighborhood of the final state* z_1 *at time* $\tau > 0$ *is given by*

$$u_\alpha(t) = B_\omega^* T^*(\tau - t)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0),$$

and the error of this approximation E_α *is given by*

$$E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0).$$

Proof . It is enough to show that $\overline{\text{Rang}(G)} = Z$ or $Ker(G^*) = \{0\}$. To this end, we observe that $B_\omega = B_\omega^*$ and $T^*(t) = T(t)$. Suppose that

$$B_\omega^* T^*(t)z = 0, \quad \forall t \in [0, \tau].$$

Then,

$$\begin{aligned} B_\omega^* T^*(t)z &= \sum_{j=1}^{\infty} e^{-\lambda_j t} B_\omega^* E_j z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > 1_\omega \phi_{j,k} = 0. \\ \iff & \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > 1_\omega \phi_{j,k}(x) = 0, \quad \forall x \in \omega. \end{aligned}$$

Hence, from Lemma 1.1, we obtain that

$$E_j z(x) = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}(x) = 0, \quad \forall x \in \omega, \quad j = 1, 2, 3, \ldots.$$

Now, putting $f(x) = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}(x), \quad \forall x \in \Omega$, we obtain that

$$\begin{cases} (\Delta + \lambda_j I)f \equiv 0 \quad \text{in} \quad \Omega, \\ f(x) = 0 \quad \forall x \in \omega. \end{cases}$$

Then, from the classical Unique Continuation for Elliptic Equations (see [18]), it follows that $f(x) = 0, \quad \forall x \in \Omega$. So,

$$\sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}(x) = 0, \quad \forall x \in \Omega.$$

On the other hand, $\{\phi_{j,k}\}$is a complete orthonormal set in $Z = L_2(\Omega)$, which implies that $< z, \phi_{j,k} >= 0$.

Therefore, $E_j z = 0, \quad j = 1, 2, 3, \ldots$, which implies that $z = 0$. So, $\overline{\text{Rang}(G)} = Z$. Hence, the system (3.1) is approximately controllable on $[0, \tau]$, and the remainder of the proof follows from Lemma 3.1. ∎

Lemma 3.2 *Let S be any dense subspace of $L_2(0, \tau; U)$. Then, system (3.1) is approximately controllable with control $u \in L_2(0, \tau; U)$ if, and only if, it is approximately controllable with control $u \in S$. i.e.,*

$$\overline{\text{Rang}(G)} = Z \iff \overline{\text{Rang}(G|_S)} = Z,$$

where $G|_S$ is the restriction of G to S.

Proof ($\Rightarrow$) Suppose $\overline{\text{Rang}(G)} = Z$ and $\overline{S} = L_2(0, \tau; U)$. Then, for a given $\epsilon > 0$ and $z \in Z$ there exits $u \in L_2(0, \tau; U)$ and a sequence $\{u_n\}_{n \geq 1} \subset S$ such that

$$\|Gu - z\| < \frac{\epsilon}{2} \quad \text{and} \quad \lim_{n \to \infty} u_n = u.$$

Therefore, $\lim_{n \to \infty} Gu_n = Gu$ and $\|Gu_n - z\| < \epsilon$ for n big enough. Hence, $\overline{\text{Rang}(G|_S)} = Z$.

($\Leftarrow$) This side is trivial. ∎

Remark 3.3 *According to the previous lemma, if the system is controllable, it is approximately controllable with control functions in the following dense spaces of* $L_2(0,\tau;U)$*:*

$$S = C([0,\tau];U), \quad S = C^{\infty}(0,\tau;U), \quad S = PC(J).$$

Moreover, the operators G, $\mathcal{W}$ *and* Γ *are well define in the space of continuous functions:* $G : C([0,\tau];U) \longrightarrow Z$ *by*

$$Gu = \int_0^{\tau} T(\tau - s)B_{\omega}u(s)ds, \tag{3.9}$$

and $G^* : Z \longrightarrow C([0,\tau];U)$ *by*

$$(G^*z)(s) = B^*(s)T^*(\tau - s)z, \quad \forall s \in [0,\tau]. \quad \forall z \in Z. \tag{3.10}$$

Also, the Controllability Grammian operator still the same $\mathcal{W} : Z \to Z$

$$\mathcal{W}z = GG^*z = \int_0^{\tau} T(\tau - s)B_{\omega}B_{\omega}^*(s)T^*(\tau - s)zds. \tag{3.11}$$

Finally, the operators $\Gamma_{\alpha} : Z \to C([0,\tau];U)$ *defined for* $0 < \alpha \leq 1$ *by*

$$\Gamma_{\alpha}z = B_{\omega}^*T^*(\tau - \cdot)(\alpha I + \mathcal{W})^{-1}z = G^*(\alpha I + GG^*)^{-1}z, \tag{3.12}$$

is an approximate inverse for the right of the operator G *in the sense that*

$$\lim_{\alpha \to 0} G\Gamma_{\alpha} = I. \tag{3.13}$$

4 Controllability of the Semilinear System

In this section we shall prove the main result of this paper, the interior approximate controllability of the Semilinear Impulsive Heat Equation given by (1.1), which is equivalent to prove the approximate controllability of the system (2.4). To this end, for all $z_0 \in Z$ and $u \in C([0,\tau];U)$ the initial value problem

$$\begin{cases} z' = -Az + B_{\omega}u + f^e(t,z,u), & t \in (0,\tau], t \neq t_k, \quad z \in Z \\ z(0) = z_0, & \\ z(t_k^+) = z(t_k^-) + I_k^e(t, z(t_k), u(t_k)), & k = 1,2,3,\ldots,p. \end{cases} \tag{4.1}$$

admits only one mild solution given by

$$\begin{aligned} z_u(t) &= T(t)z_0 + \int_0^t T(t-s)B_{\omega}u(s)ds \qquad (4.2)\\ &+ \int_0^t T(t-s)f^e(s, z_u(s), u(s))ds \qquad (4.3)\\ &+ \sum_{0<t_k<t} T(t-t_k)I_k^e(t_k, z(t_k), u(t_k)), \quad t \in [0,\tau]. \end{aligned}$$

Now, we are ready to present and prove the main result of this paper, which is the interior approximate controllability of the semilinear impulsive heat equation (1.1). We shall define the operator $\mathcal{K}^\alpha : PC([0,\tau];Z) \times C([0,\tau];U) \to PC([0,\tau];Z) \times C([0,\tau];U)$ by the following formula:

$$(y,v) = (\mathcal{K}_1^\alpha(z,u), \mathcal{K}_2^\alpha(z,u)) = \mathcal{K}^\alpha(z,u)$$

where

$$\begin{aligned} y(t) &= \mathcal{K}_1^\alpha(z,u)(t) = T(t)z_0 + \int_0^t T(t-s)B_\omega(\Gamma_\alpha \mathcal{L}(z,u))(s)ds \\ &+ \int_0^t T(t-s)f^e(s,z(s),u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k^e(t_k,z(t_k),u(t_k)), \end{aligned} \tag{4.4}$$

and

$$v(t) = \mathcal{K}_2^\alpha(z,u)(t) = (\Gamma_\alpha \mathcal{L}(z,u))(t) = B_\omega^* T^*(\tau-t)(\alpha I + \mathcal{W})^{-1}\mathcal{L}(z,u), \tag{4.5}$$

with $\mathcal{L} : PC([0,\tau];Z) \times C([0,\tau];U) \to Z$ is given by

$$\begin{aligned} \mathcal{L}(z,u) &= z_1 - T(\tau)z_0 - \int_0^\tau T(\tau-s)f^e(s,z(s),u(s))ds \\ &- \sum_{0<t_k<\tau} T(\tau-t_k)I_k^e(t_k,z(t_k),u(t_k)). \end{aligned} \tag{4.6}$$

Theorem 4.1 *The nonlinear system (1.1) is approximately controllable on* $[0,\tau]$*. Moreover, a sequence of controls steering the system (1.1) from initial state* z_0 *to an* ϵ*-neighborhood of the final state* z_1 *at time* $\tau > 0$ *is given by*

$$u_\alpha(t) = B_\omega^* T^*(\tau-t)(\alpha I + \mathcal{W})^{-1}\mathcal{L}(z_\alpha, u_\alpha),$$

and the error of this approximation $E_\alpha z$ *is given by*

$$E_\alpha z = \alpha(\alpha I + \mathcal{W})^{-1}\mathcal{L}(z_\alpha, u_\alpha),$$

where

$$\begin{aligned} z_\alpha(t) &= T(t)z_0 + \int_0^t T(t-s)B_\omega u_\alpha(s)ds \\ &+ \int_0^t T(t-s)f^e(s,z_\alpha(s),u_\alpha(s))ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k^e(t_k,z_\alpha(t_k),u_\alpha(t_k)), \quad t \in [0,\tau]. \end{aligned}$$

Proof We shall prove this Theorem by claims. Before we note that $\|B_\omega\| = 1$ and $\|T(t)\| \le e^{-\lambda_1 t}, \quad t \ge 0.$

Claim 1. The operator $\mathcal{K}^\alpha$ is continuous. In fact, it is enough to prove that the operators:

$$\mathcal{K}_1^\alpha : PC([0,\tau];Z) \times C([0,\tau];U) \to PC([0,\tau];Z)$$

and

$$\mathcal{K}_2^\alpha : PC([0,\tau];Z) \times C([0,\tau];U) \to C([0,\tau];U),$$

define above are continuous. The continuity of $\mathcal{K}_1^\alpha$ follows from the continuity of the nonlinear functions $f^\alpha(t,z,u)$, $I_k^e(t,z,u)$ and the following estimate

$$\begin{aligned}\|\mathcal{K}_1^\alpha(z,u)(t) - \mathcal{K}_1^\alpha(w,v)(t)\| &\leq \int_0^t e^{-\lambda_1(t-s)}\|(\alpha I + \mathcal{W})^{-1}\|\|\mathcal{L}(z,u) - \mathcal{L}(w,v)\|ds \\ &+ \int_0^t e^{-\lambda_1(t-s)}\|f^e(s,z(s),u(s)) - f^e(s,w(s),v(s))\|ds \\ &+ \sum_{0<t_k<t} e^{-\lambda_1(t-t_k)}\|I_k^e(t_k,z(t_k),u(t_k)) - I_k^e(t_k,w(t_k),v(t_k))\|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\mathcal{L}(z,u) - \mathcal{L}(w,v)\| &\leq \tau \sup_{s\in[0,\tau]} \|f^e(s,z(s),u(s)) - f^e(s,w(s),v(s))\| \\ &+ \sum_{0<t_k<\tau} e^{-\lambda_1(t-t_k)}\|I_k^e(t_k,z(t_k),u(t_k)) - I_k^e(t_k,w(t_k),v(t_k))\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathcal{K}_1^\alpha(z,u) - \mathcal{K}_1^\alpha(w,v)\| &\leq L_1 \sup_{s\in[0,\tau]} \|f^e(s,z(s),u(s)) - f^e(s,w(s),v(s))\| \\ &+ L_2 \sum_{0<t_k<\tau} \|I_k^e(t_k,z(t_k),u(t_k)) - I_k^e(t_k,w(t_k),v(t_k))\|.\end{aligned}$$

where $L_1 = \tau(\tau\|(\alpha I + \mathcal{W})^{-1}\| + 1)$ and $L_2 = (1 + \tau\|(\alpha I + \mathcal{W})^{-1}\|)$.
The continuity of the operator $\mathcal{K}_2^\alpha$ follows from the continuity of the operators $\mathcal{L}$ and Γ_α define above.
Claim 2. The operator $\mathcal{K}^\alpha$ is compact. In fact, let D be a bounded subset of $PC(J;Z) \times C(J;U)$. It follows that $\forall (z,u) \in D$,we have

$$\|f^e(\cdot,z,u)\| \leq L_3, \qquad \|(\alpha I + \mathcal{W})^{-1}\mathcal{L}(z,u)\| \leq L_4,$$
$$\|\mathcal{L}(z,u)\| \leq L_5, \quad \|I_k^e(\cdot,z,u)\| \leq l_k, \quad k = 1,2,\ldots,p.$$

Therefore , $\mathcal{K}(D)$ is uniformly bounded.
Now, consider the following estimate:

$$\begin{aligned}|||\mathcal{K}^\alpha(z,u)(t_2) - \mathcal{K}^\alpha(z,u)(t_1)||| &= \|\mathcal{K}_1^\alpha(z,u)(t_2) - \mathcal{K}_1^\alpha(z,u)(t_1)\| \\ &+ \|\mathcal{K}_2^\alpha(z,u)(t_2) - \mathcal{K}_2^\alpha(z,u)(t_1)\|.\end{aligned}$$

Without lose of generality we assume that $0 < t_1 < t_2$. On the other hand we have:

$$
\begin{aligned}
\|\mathcal{K}_1^\alpha(z,u)(t_2) - \mathcal{K}_1^\alpha(z,u)(t_1)\| &\leq \|T(t_2) - T(t_1)\|\|z_0\| \\
&+ \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\|\|\mathcal{L}(z,u)(s)\|ds \\
&+ \int_{t_1}^{t_2} \|T(t_2 - s)\|\|\mathcal{L}(z,u)(s)\|ds \\
&+ \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\|\|f^e(s, z(s), u(s))\|ds \\
&+ \int_{t_1}^{t_2} \|T(t_2 - s)\|\|f^e(s, z(s), u(s))\|ds \\
&+ \sum_{0<t_k<t_1} \|T(t_2 - t_k) - T(t_1 - t_k)\|\|I_k^e(t_k, z(t_k), u(t_k))\| \\
&+ \sum_{t_1<t_k<t_2} \|T(t_2 - t_k)I_k^e(t_k, z(t_k), u(t_k))\|,
\end{aligned}
$$

and

$$
\|\mathcal{K}_2^\alpha(z,u)(t_2) - \mathcal{K}_2^\alpha(z,u)(t_1)\| \leq \|T^*(\tau - t_2) - T^*(\tau - t_1)\|\|(\alpha I + \mathcal{W})^{-1}\mathcal{L}(z,u)\|.
$$

On the other hand, since $T(t)$ is a compact operator for $t > 0$, then from [17] we know that the function $0 < t \to T(t)$ is uniformly continuous. So,

$$
\lim_{|t_2 - t_1| \to 0} \|T(t_2) - T(t_1)\| = 0.
$$

Consequently, if we take a sequence $\{\phi_j : j = 1, 2, \dots\}$ on $\mathcal{K}^\alpha(D)$, this sequence is uniformly bounded and equicontinuous on the interval $[0, t_1]$ and, by Arzela theorem, there is a subsequence $\{\phi_j^1 : j = 1, 2, \dots\}$ of $\{\phi_j : j = 1, 2, \dots\}$, which is uniformly convergent on $[0, t_1]$.

Consider the sequence $\{\phi_j^1 : j = 1, 2, \dots\}$ on the interval $(t_1, t_2]$. On this interval the sequence $\{\phi_j^1 : j = 1, 2, \dots\}$ is uniformly bounded and equicontinuous, and for the same reason, it has a subsequence $\{\phi_j^2\}$ uniformly convergent on $[0, t_2]$.

Continuing this process for the intervals $(t_2, t_3]$, $(t_3, t_4]$, $\dots$, $(t_p, \tau]$, we see that the sequence $\{\phi_j^{p+1} : j = 1, 2, \dots\}$ converges uniformly on the interval $[0, \tau]$. This means that $\overline{\mathcal{K}^\alpha(D)}$ is compact, which implies that the operator $\mathcal{K}^\alpha$ is compact.

Claim 3.

$$
\lim_{|||(z,u)||| \to \infty} \frac{|||\mathcal{K}^\alpha(z,u)|||}{|||(z,u)|||} = 0,
$$

where $|||(z,u)||| = \|z\| + \|u\|$ is the norm in the space $PC([0,\tau]; Z) \times C(0,\tau; Z)$. In fact, consider the following estimates:

$$
\|\mathcal{L}(z,u)\| \leq M_1 + M_2\{\overline{a}_0\|z\|^{\alpha_0} + \overline{b}_0\|u\|^{\beta_0} + \overline{c}_0\} + M_3 \sum_{0<t_k<\tau} \{\overline{a}_k\|z\|^{\alpha_k} + \overline{b}_k\|u\|^{\beta_k} + \overline{c}_k\},
$$

where

$$M_1 = \|z_1\| + e^{-\lambda_1\tau}\|z_0\|, \quad M_2 = \frac{1}{-\lambda_1}(e^{-\lambda_1\tau} - 1) \quad \text{and} \quad M_3 = e^{-\lambda_1\tau}.$$

$$\begin{aligned}\|\mathcal{K}_2^\alpha(z,u)\| \le \quad & M_3M_1\|(\alpha I + \mathcal{W})^{-1}\| + M_3M_2\|(\alpha I + \mathcal{W})^{-1}\|\{\overline{a}_0\|z\|^{\alpha_0} + \overline{b}_0\|u\|^{\beta_0} + \overline{c}_0\} \\ + \quad & M_3M_2\|(\alpha I + \mathcal{W})^{-1}\| \sum_{0<t_k<\tau}\{\overline{a}_k\|z\|^{\alpha_k} + \overline{b}_k\|u\|^{\beta_k} + \overline{c}_k\}.\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{K}_1^\alpha(z,u)\| \le & M_3\{\|z_0\| + M_1M_2\|(\alpha I + \mathcal{W})^{-1}\|\} \\ & + M_2\{1 + M_2M_3\|(\alpha I + \mathcal{W})^{-1}\|\}\{\overline{a}_0\|z\|^{\alpha_0} + \overline{b}_0\|u\|^{\beta_0} + c_0\} \\ & + M_3\{1 + M_2M_3\|(\alpha I + \mathcal{W})^{-1}\|\} \sum_{0<t_k<\tau}\{\overline{a}_k\|z\|^{\alpha_k} + \overline{b}_k\|u\|^{\beta_k} + \overline{c}_k\}.\end{aligned}$$

Therefore,

$$\begin{aligned}|||\mathcal{K}^\alpha(z,u)||| = \|\mathcal{K}_1^\alpha(z,u)\| + \|\mathcal{K}_2^\alpha(z,u)\| \le M_4 & \\ + \{M_3M_2\|(\alpha I + \mathcal{W})^{-1}\|\{1 + 2M_2\}\{\overline{a}_0\|z\|^{\alpha_0} + \overline{b}_0\|u\|^{\beta_0} + \overline{c}_0\} & \\ + \{M_3M_2\|(\alpha I + \mathcal{W})^{-1}\|\{1 + M_3\} + M_3\} \sum_{0<t_k<\tau}\{\overline{a}_k\|z\|^{\alpha_k} + \overline{b}_k\|u\|^{\beta_k} + \overline{c}_k\}, &\end{aligned}$$

where M_4 is given by:

$$M_4 = M_3\{\|z_0\| + (M_2 + 1)M_1\|(\alpha I + \mathcal{W})^{-1}\|\}.$$

Hence

$$\begin{aligned}\frac{|||\mathcal{K}^\alpha(z,u)|||}{|||(z,u)|||} \quad \le \quad & \frac{M_4}{\|z\| + \|u\|} \\ + \quad & \{M_3M_2\|(\alpha I + \mathcal{W})^{-1}\|\{1 + M_2\}\} \\ & \times\{\overline{a}_0\|z\|^{\alpha_0 - 1} + \overline{b}_0\|u\|^{\beta_0 - 1} + \frac{\overline{c}_0}{\|z\| + \|u\|}\} \\ + \quad & \{M_3M_2\|(\alpha I + \mathcal{W})^{-1}\|\{1 + M_3\} + M_3\} \times \\ & \sum_{0<t_k<\tau}\{\overline{a}_k\|z\|^{\alpha_k - 1} + \overline{b}_k\|u\|^{\beta_k - 1} + \frac{\overline{c}_k}{\|z\| + \|u\|}\},\end{aligned}$$

and

$$\lim_{|||(z,u)|||\to\infty} \frac{|||\mathcal{K}^\alpha(z,u)|||}{|||(z,u)|||} = 0. \tag{4.7}$$

Claim 4. The operator $\mathcal{K}^\alpha$ has a fixed point. In fact, for a fixed $0 < \rho < 1$, there exists $R > 0$ big enough such that

$$|||\mathcal{K}^\alpha(z,u)||| \le \rho|||(z,u)|||, \quad |||(z,u)||| = R.$$

Hence, if we denote by $B(0,R)$ the ball of center zero and radius $R>0$, we get that $\mathcal{K}^\alpha(\partial B(0,R)) \subset B(0,R)$. Since $\mathcal{K}^\alpha$ is compact and maps the sphere $\partial B(0,R)$ into the interior of the ball $B(0,R)$, we can apply Rothe's fixed point Theorem 1.1 to ensure the existence of a fixed point $(z_\alpha, u_\alpha) \in B(0,R) \subset PC([0,\tau];Z) \times C([0,\tau];U)$ such that

$$(z_\alpha, u_\alpha) = \mathcal{K}^\alpha(z_\alpha, u_\alpha). \tag{4.8}$$

Claim 5. The sequence $\{(z_\alpha, u_\alpha)\}_{\alpha\in(0,1]}$ is bounded. In fact, for the purpose of contradiction, let us assume that $\{(z_\alpha, u_\alpha)\}_{\alpha\in(0,1]}$ is unbounded. Then, there exits a subsequence $\{(z_{\alpha_n}, u_{\alpha_n})\}_{\alpha\in(0,1]} \subset \{(z_\alpha, u_\alpha)\}_{\alpha\in(0,1]}$ such that

$$\lim_{n\to\infty} |||(z_{\alpha_n}, u_{\alpha_n})||| = \infty.$$

On the other hand, from (4.7) we know for all $\alpha \in (0,1]$ that

$$\lim_{n\to\infty} \frac{|||\mathcal{K}^\alpha(z_{\alpha_n}, u_{\alpha_n})|||}{|||(z_{\alpha_n}, u_{\alpha_n})|||} = 0.$$

Particularly, we have the following situation:

$$\begin{array}{cccccc}
\frac{|||\mathcal{K}^{\alpha_1}(z_{\alpha_1},u_{\alpha_1})|||}{|||(z_{\alpha_1},u_{\alpha_1})|||} & \frac{|||\mathcal{K}^{\alpha_1}(z_{\alpha_2},u_{\alpha_2})|||}{|||(z_{\alpha_2},u_{\alpha_2})|||} & \frac{|||\mathcal{K}^{\alpha_1}(z_{\alpha_3},u_{\alpha_3})|||}{|||(z_{\alpha_3},u_{\alpha_3})|||} & \cdots\cdots & \frac{|||\mathcal{K}^{\alpha_1}(z_{\alpha_n},u_{\alpha_n})|||}{|||(z_{\alpha_n},u_{\alpha_n})|||} & \to 0. \\
\frac{|||\mathcal{K}^{\alpha_2}(z_{\alpha_1},u_{\alpha_1})|||}{|||(z_{\alpha_1},u_{\alpha_1})|||} & \frac{|||\mathcal{K}^{\alpha_2}(z_{\alpha_2},u_{\alpha_2})|||}{|||(z_{\alpha_2},u_{\alpha_2})|||} & \frac{|||\mathcal{K}^{\alpha_2}(z_{\alpha_3},u_{\alpha_3})|||}{|||(z_{\alpha_3},u_{\alpha_3})|||} & \cdots\cdots & \frac{|||\mathcal{K}^{\alpha_2}(z_{\alpha_n},u_{\alpha_n})|||}{|||(z_{\alpha_n},u_{\alpha_n})|||} & \to 0. \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\frac{|||\mathcal{K}^{\alpha_k}(z_{\alpha_1},u_{\alpha_1})|||}{|||(z_{\alpha_1},u_{\alpha_1})|||} & \frac{|||\mathcal{K}^{\alpha_k}(z_{\alpha_2},u_{\alpha_2})|||}{|||(z_{\alpha_2},u_{\alpha_2})|||} & \frac{|||\mathcal{K}^{\alpha_k}(z_{\alpha_3},u_{\alpha_3})|||}{|||(z_{\alpha_3},u_{\alpha_3})|||} & \cdots\cdots & \frac{|||\mathcal{K}^{\alpha_k}(z_{\alpha_n},u_{\alpha_n})|||}{|||(z_{\alpha_n},u_{\alpha_n})|||} & \to 0.
\end{array}$$

Now, applying Cantor's diagonalization process, we obtain that

$$\lim_{n\to\infty} \frac{|||\mathcal{K}^{\alpha_n}(z_{\alpha_n}, u_{\alpha_n})|||}{|||(z_{\alpha_n}, u_{\alpha_n})|||} = 0,$$

and from (4.8) we have that

$$\frac{|||\mathcal{K}^{\alpha_n}(z_{\alpha_n}, u_{\alpha_n})|||}{|||(z_{\alpha_n}, u_{\alpha_n})|||} = 1,$$

which is evidently a contradiction. Then, the claim is true and there exists $\gamma > 0$ such that

$$|||(z_{\alpha_n}, u_{\alpha_n})||| \le \gamma, \quad (0 < \alpha \le 1).$$

Therefore, without loss of generality, we can assume that the sequence $\mathcal{L}(z_\alpha, u_\alpha)$ converges to $y \in Z$. So, if

$$u_\alpha = \Gamma_\alpha \mathcal{L}(z_\alpha, u_\alpha) = G^*(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha).$$

Then,

$$\begin{aligned} Gu_\alpha &= G\Gamma_\alpha \mathcal{L}(z_\alpha, u_\alpha) = GG^*(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha) \\ &= (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha) \\ &= \mathcal{L}(z_\alpha, u_\alpha) - \alpha(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha). \end{aligned}$$

Hence,

$$Gu_\alpha - \mathcal{L}(z_\alpha, u_\alpha) = -\alpha(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha).$$

To conclude the proof of this Theorem, it enough to prove that

$$\lim_{\alpha\to 0}\{-\alpha(\alpha I + GG^*)^{-1}\}\mathcal{L}(z_\alpha, u_\alpha) = 0.$$

From Lemma 3.1.d) we get that

$$\begin{aligned} \lim_{\alpha\to 0}\{\alpha(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha)\} &= \lim_{\alpha\to 0}\alpha(\alpha I + GG^*)^{-1}y \\ &+ \lim_{\alpha\to 0}\alpha(\alpha I + GG^*)^{-1}(\mathcal{L}(z_\alpha, u_\alpha) - y) \\ &= \lim_{\alpha\to 0} -\alpha(\alpha I + GG^*)^{-1}(\mathcal{L}(z_\alpha, u_\alpha) - y) \end{aligned}$$

On the other hand, from Proposition 3.1, we get that

$$\|\alpha(\alpha I + GG^*)^{-1}(\mathcal{L}(z_\alpha, u_\alpha) - y)\| \le \|\mathcal{L}(z_\alpha, u_\alpha) - y)\|.$$

Therefore, since $\mathcal{L}(z_\alpha, u_\alpha)$ converges to y, we get that

$$\lim_{\alpha\to 0}\{-\alpha(\alpha I + GG^*)^{-1}(\mathcal{L}(z_\alpha, u_\alpha) - y)\} = 0.$$

Consequently,

$$\lim_{\alpha\to 0}\{-\alpha(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha)\} = 0.$$

Then,

$$\lim_{\alpha\to 0}\{Gu_\alpha - \mathcal{L}(z_\alpha, u_\alpha)\} = 0.$$

Therefore,

$$\begin{aligned} \lim_{\alpha\to 0}\{T(\tau)z_0 + \int_0^\tau T(\tau - s)B_\omega u_\alpha(s)ds &+ \int_0^\tau T(\tau - s)f^e(s, z_\alpha(s), u_\alpha(s))ds \\ &+ \sum_{0<t_k<\tau} T(\tau - t_k)I_k^e(z_\alpha(t_k), u_\alpha(t_k))\} = z_1, \end{aligned}$$

and the proof of the theorem is completed.

∎

As a consequence of the foregoing theorem we can prove the following characterization:

Theorem 4.2 *The Impulsive Semilinear System (1.1) is approximately controllable if for all states z_0 and a final state z_1 and $\alpha \in (0, 1]$ the operator $\mathcal{K}^\alpha$ given by (4.4)-(4.6) has a fixed point and the sequence $\{\mathcal{L}(z_\alpha, u_\alpha)\}_{\alpha\in(0,1]}$ converges. i.e.,*

$$(z_\alpha, u_\alpha) = \mathcal{K}^\alpha(z_\alpha, u_\alpha),$$

$$\lim_{\alpha\to 0}\mathcal{L}(z_\alpha, u_\alpha) = y \in Z.$$

5 Final Remark

Our technique is simple and can be apply to those system involving compact semigroups like some control system governed by diffusion processes. For example, the Benjamin -Bona-Mohany Equation, the strongly damped wave equations, beam equations, etc.

Example 5.1 *The original Benjamin -Bona-Mohany Equation is a non-linear one, in [16] the authors proved the approximate controllability of the linear part of this equation, which is the fundamental base for the study of the controllability of the non linear BBM equation. So, our next work is concerned with the controllability of non linear BBM equation*

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x) + f(t,z,u(t)), \quad t \in (0,\tau), \qquad x \in \Omega, \\ z(t,x) = 0, \quad t \geq 0, \qquad x \in \partial\Omega, \\ z(0,x) = z_0(x), x \in \Omega, \\ z(t_k^+,x) = z(t_k^-,x) + I_k(t, z(t_k,x), u(t_k,x)), x \in \Omega, \end{cases}$$

where $a \geq 0$ and $b > 0$ are constants, $k = 1,2,\ldots,p$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, $z_0 \in L_2(\Omega)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω,the distributed control u belongs to $C([0,\tau]; L_2(\Omega;))$ and $f, I_k \in C([0,\tau] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, $k = 1,2,3,\ldots,p$.

Example 5.2 *We believe that this technique can be applied to prove the interior controllability of the strongly damped wave equation with Dirichlet boundary conditions*

$$\begin{cases} w_{tt} + \eta(-\Delta)^{1/2} w_t + \gamma(-\Delta) w = 1_\omega u(t,x) + f(t, w, w_t, u(t)), & in \quad (0,\tau) \times \Omega, \\ w = 0, & in \quad (0,\tau) \times \partial\Omega, \\ w(0,x) = w_0(x), \quad w_t(0,x) = w_1(x), & in \quad \Omega, \\ w(t_k^+,x) = w(t_k^-,x) + I_k^1(t, w(t_k,x), w_t(t_k,x), u(t_k,x)), \quad x \in \Omega, & \\ w_t(t_k^+,x) = w_t(t_k^-,x) + I_k^2(t, w(t_k,x), w_t(t_k,x), u(t_k,x)), \quad x \in \Omega, & \end{cases}$$

in the space $Z_{1/2} = D((-\Delta)^{1/2}) \times L_2(\Omega)$, $k = 1,2,\ldots,p$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, , ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω,the distributed control $u \in C([0,\tau]; L_2(\Omega))$, η, γ are positive numbers and $f, I_k^1, I_k^2 \in C([0,\tau] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, $k = 1,2,3,\ldots,p$.

Example 5.3 *Another example where this technique may be applied is a partial differential equations modeling the structural damped vibrations of a string or a beam:*

$$\begin{cases} y_{tt} - 2\beta\Delta y_t + \Delta^2 y = 1_\omega u(t,x) + f(t, y, y_t, u(t)), & on \quad (0,\tau) \times \Omega, \\ y = \Delta y = 0, & on \quad (0,\tau) \times \partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), & in \quad \Omega, \\ y(t_k^+,x) = y(t_k^-,x) + I_k^1(t, y(t_k,x), y_t(t_k,x), u(t_k,x)), x \in \Omega, & \\ y_t(t_k^+,x) = y_t(t_k^-,x) + I_k^2(t, y(t_k,x), y_t(t_k,x), u(t_k,x)), x \in \Omega, & \end{cases}$$

where Ω is a bounded domain in $I\!R^n$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in C([0,\tau]; L_2(\Omega))$ and $y_0 \in H^2(\Omega) \cap H_0^1, y_1 \in L_2(\Omega)$.

Moreover, our result can be formulated in a more general setting. Indeed, we can consider the following semilinear evolution equation in a general Hilbert space Z

$$\begin{cases} \acute{z} = -Az + Bu(t) + f^e(t,z,u), \quad z \in Z, \quad t \in (0,\tau], \\ z(0) = z_0, \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k), u(t_k)), k = 1,2,3,\ldots,p. \end{cases} \tag{5.1}$$

where $u \in C([0,\tau];U)$, $U = Z$, $B_\omega : U \longrightarrow Z$, $B_\omega u = 1_\omega u$ is a bounded linear operator,$I_k^e, f^e : [0,\tau] \times Z \times U \to Z$, $A : D(A) \subset Z \to Z$ is an unbounded linear operator in Z with the following spectral decomposition:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k},$$

with the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \cdots \lambda_n \to \infty$ of A having finite multiplicity γ_j equal to the dimension of the corresponding eigenspaces, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenfunctions of A. The operator $-A$ generates a strongly continuous compact semigroup $\{T_A(t)\}_{t\geq 0}$ given by

$$T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}.$$

The control $u \in C([0,\tau];U)$, with $U = Z$, $B : Z \to Z$ is a linear and bounded operator(linear and continuous) and the functions $f^e, I_k^e : [0,\tau] \times Z \times U \to Z$ are smooth enough and

$$\|f^e(t,z,u)\|_Z \leq \tilde{a}_0 \|z\|_Z^{\alpha_0} + \tilde{b}_0 \|u\|_Z^{\beta_0} + \tilde{c}_0 \tag{5.2}$$

$$\|I_k^e(t,z,u)\|_Z \leq \tilde{a}_k \|z\|_Z^{\alpha_k} + \tilde{b}_k \|u\|_Z^{\beta_k} + \tilde{c}_k, k = 1,2,3,\ldots,p. \tag{5.3}$$

In this case the characteristic function set is a particular operator B, and the following theorem is a generalization of Theorem 4.1.

Theorem 5.1 *If vectors $B^*\phi_{j,k}$ are linearly independent in Z, then the system (5.1) is approximately controllable on* $[0,\tau]$.

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DOI: 10.7862/rf.2015.8

Hugo Leiva - corresponding author
email: `hleiva@ula.ve`
Universidad de los Andes
Facultad de Ciencias, Departamentode Matemática
Mérida 5101-Venezuela

Nelson Merentes
email: `nmerucv@gmail.com`
Universidad Central de Venezuela
Facultad de Ciencias, Departamento de Matemática
Caracas -Venezuela

Received 27.03.2014