Starlikeness and convexity of certain integral operators defined by convolution

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Abstract: We define two new general integral operators for certain analytic functions in the unit disc $U$ and give some sufficient conditions for these integral operators on some subclasses of analytic functions.

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1 Introduction

Let $A_p(n)$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \ (p,n \in \mathbb{N} = \{1, 2, 3, \ldots\}). \quad (1.1)$$

which is analytic in open unit disc $U = \{z \in \mathbb{C} | |z| < 1\}$. In particular, we set $A_p(1) = A_p, A_1(1) = A_1 := A$.

If $f \in A_p(n)$ is given by (1.1) and $g \in A_p(n)$ is given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k \ (p,n \in \{1, 2, 3, \ldots\}). \quad (1.2)$$

then the Hadamard product (or convolution) $f * g$ of $f$ and $g$ is given by

$$(f * g)(z) = z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k = (g * f)(z). \quad (1.3)$$
We observe that several known operators are deducible from the convolutions. That is, for various choices of \( g \) in (1.3), we obtain some interesting operators. For example, for functions \( f \in \mathcal{A}_p(n) \) and the function \( g \) is defined by

\[
g(z) = z^p + \sum_{k=p+n}^{\infty} \psi_{k,m}(\alpha, \lambda, l, p)z^k \quad (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \tag{1.4}
\]

where

\[
\psi_{k,m}(\alpha, \lambda, l, p) = \left[ \frac{\Gamma(k+1)\Gamma(p-\alpha+1)}{\Gamma(p+1)\Gamma(k-\alpha+1)} \right]^m \Gamma(p+1)\Gamma(k-\alpha+1)^{p+l} \quad (m \in \mathbb{N}_0) \tag{1.4}
\]

The convolution (1.3) with the function \( g \) is defined by (1.4) gives an operator studied by Bulut ([1]).

\[
(f * g)(z) = D_{\lambda, p}^{m, \alpha} f(z)
\]

Using convolution we introduce the new classes \( \mathcal{US}_p^g(\delta, \beta, b) \) and \( \mathcal{UK}_p^g(\delta, \beta, b) \) as follows

**Definition 1.1** A functions \( f \in \mathcal{A}_p(n) \) is in the class \( \mathcal{US}_p^g(\delta, \beta, b) \) if and only if \( f \) satisfies

\[
\text{Re}\left\{ p + \frac{1}{b} \left( \frac{z(f * g)(z)'}{(f * g)(z)} - p \right) \right\} > \delta \left| \frac{1}{b} \left( \frac{z(f * g)(z)'}{(f * g)(z)} - p \right) \right| + \beta, \tag{1.5}
\]

where \( z \in \mathbb{U}, b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p. \)

**Definition 1.2** A functions \( f \in \mathcal{A}_p(n) \) is in the class \( \mathcal{US}_p^g(\delta, \beta, b) \) if and only if \( f \) satisfies

\[
\text{Re}\left\{ p + \frac{1}{b} \left( 1 + \frac{z(f * g)(z)''}{(f * g)(z)'} - p \right) \right\} > \delta \left| \frac{1}{b} \left( 1 + \frac{z(f * g)(z)''}{(f * g)(z)'} - p \right) \right| + \beta, \tag{1.6}
\]

where \( z \in \mathbb{U}, b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p. \)

Note that

\[
f \in \mathcal{UK}_p^g(\delta, \beta, b) \iff \frac{zf'(z)}{p} \in \mathcal{US}_p^g(\delta, \beta, b).
\]

**Remark 1.1** (i) For \( \delta = 0 \), we have

\[
\mathcal{UK}_p^g(0, \beta, b) = \mathcal{K}_p^g(\beta, b) \quad \mathcal{US}_p^g(0, \beta, b) = \mathcal{S}_p^g(\beta, b)
\]

(ii) For \( \delta = 0 \) and \( \beta = 0 \)

\[
\mathcal{UK}_p^g(0, 0, b) = \mathcal{K}_p^g(b) \quad \mathcal{US}_p^g(0, 0, b) = \mathcal{S}_p^g(b)
\]

(iii) For \( \delta = 0, \beta = 0 \) and \( b = 1 \)

\[
\mathcal{UK}_p^g(0, 0, b) = \mathcal{K}_p^g \quad \mathcal{US}_p^g(0, 0, b) = \mathcal{S}_p^g
\]

(iv) For \( (f_j * g)(z) = D_{\lambda, p}^{m, \alpha} f_j(z) \), we have two classes \( \mathcal{UK}_p^{n, j, p, n}(\delta_j, \beta_j, b) \) and \( \mathcal{US}_{\alpha, \lambda, p, n}(\delta_j, \beta_j, b) \) which is introduced by Guney and Bulut [1].
Definition 1.3 Let $\eta \in N, m = (m_1, \ldots, m_{\eta}) \in N_{0}^{\eta}$ and $k = (k_1, \ldots, k_{\eta}) \in R_{+}^{\eta}$. One defines the following general integral operators:

$$\mathcal{I}_{g}^{p,\eta,m,k} : \mathcal{A}_{p}(n)^{\eta} \rightarrow \mathcal{A}_{p}(n)$$

$$\mathcal{G}_{g}^{p,\eta,m,k} : \mathcal{A}_{p}(n)^{\eta} \rightarrow \mathcal{A}_{p}(n)$$

such that

$$\mathcal{I}_{g}^{p,\eta,m,k}(z) = \frac{z}{\pi} \int_{0}^{p} t^{p-1} \prod_{j=1}^{\eta} \left( \frac{((f_j * g)(t))}{t^{p}} \right)^{k_j} dt,$$

$$\mathcal{G}_{g}^{p,\eta,m,k}(z) = \frac{z}{\pi} \int_{0}^{p} t^{p-1} \prod_{j=1}^{\eta} \left( \frac{((f_j * g)'(t))}{t^{p-1}} \right)^{k_j} dt,$$

where $z \in U$, $f, g \in \mathcal{A}_{p}(n), 1 \leq j \leq \eta$.

Remark 1.2 (i) For $\eta = 1, m_1 = m, k_1 = k$, and $f_1 = f$, we have the new two new integral operators

$$\mathcal{I}_{g}^{p,\eta,m,k}(z) = \frac{z}{\pi} \int_{0}^{p} t^{p-1} \left( \frac{(f_1 * g)(t)}{t^{p}} \right)^{k_1} dt,$$

$$\mathcal{G}_{g}^{p,\eta,m,k}(z) = \frac{z}{\pi} \int_{0}^{p} t^{p-1} \left( \frac{(f_1 * g)'(t)}{t^{p-1}} \right)^{k_1} dt,$$

(ii) For $(f_j * g)(z) = D_{m,\lambda,\alpha}^{m,\sigma,\tau} f_j(z)$, we have

$$\mathcal{I}_{g}^{p,\eta,m,k}(z) = \frac{z}{\pi} \int_{0}^{p} t^{p-1} \prod_{j=1}^{\eta} \left( \frac{D_{m,\lambda,\alpha}^{m,\sigma,\tau} f_j(t)}{t^{p}} \right)^{k_j} dt,$$

$$\mathcal{G}_{g}^{p,\eta,m,k}(z) = \frac{z}{\pi} \int_{0}^{p} t^{p-1} \prod_{j=1}^{\eta} \left( \frac{D_{m,\lambda,\alpha}^{m,\sigma,\tau} f_j'(t)}{t^{p-1}} \right)^{k_j} dt,$$

These operators were introduced by Bulut [1].

(iii) If we take $g(z) = z^{\beta}/(1 - z)$, we have

$$\mathcal{I}_{g}^{p,\eta,m,k}(z) = \frac{z}{\pi} \int_{0}^{p} t^{p-1} \prod_{j=1}^{\eta} \left( \frac{(f_j)(t)}{t^{p}} \right)^{k_j} dt,$$

$$\mathcal{G}_{g}^{p,\eta,m,k}(z) = \frac{z}{\pi} \int_{0}^{p} t^{p-1} \prod_{j=1}^{\eta} \left( \frac{(f_j)'(t)}{t^{p-1}} \right)^{k_j} dt,$$

These two operators were introduced by Frasin [3].

2 Sufficient Conditions for $\mathcal{I}_{g}^{p,\eta,m,k}(z)$

Theorem 2.1 Let $\eta \in N, m = (m_1, \ldots, m_{\eta}) \in N_{0}^{\eta}$ and $k = (k_1, \ldots, k_{\eta}) \in R_{+}^{\eta}$. Also let $b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathcal{A}_{p}([0, \beta], \mathcal{A}_{p}(n))$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p,$$
then the integral operator $I_{g}^{p,\eta,m,k}(z)$, defined by (1.8), is in the class $K_{g}^{p}(\tau,b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_{j}(\beta_{j} - p).$$

**Proof.** From the definition (1.8), we observe that $I_{g}^{p,\eta,m,k}(z) \in A_{p}(n)$. We can easily see that

$$(I_{g}^{p,\eta,m,k}(z))' = p^{p-1} \prod_{j=1}^{\eta} \left( \frac{(f_{j} \ast g)(z)}{z^{p}} \right)^{k_{j}}. \quad (2.2)$$

Differentiating (2.2) logarithmically and multiplying by '$z$', we obtain

$$z (I_{g}^{p,\eta,m,k}(z))'' = p - 1 + \sum_{j=1}^{\eta} k_{j} \left( \frac{z((f_{j} \ast g)(z))'}{(f_{j} \ast g)(z)} - p \right) \quad (2.3)$$

or equivalently

$$1 + z \frac{z (I_{g}^{p,\eta,m,k}(z))''}{(I_{g}^{p,\eta,m,k}(z))'} - p = \sum_{j=1}^{\eta} k_{j} \left( \frac{z((f_{j} \ast g)(z))'}{(f_{j} \ast g)(z)} - p \right) \quad (2.4)$$

Then, by multiplying (2.4) with '1/b', we have

$$\frac{1}{b} \left( 1 + z \frac{z (I_{g}^{p,\eta,m,k}(z))''}{(I_{g}^{p,\eta,m,k}(z))'} - p \right) = \sum_{j=1}^{\eta} k_{j} \frac{1}{b} \left( \frac{z((f_{j} \ast g)(z))'}{(f_{j} \ast g)(z)} - p \right) \quad (2.5)$$

or

$$p + \frac{1}{b} \left( 1 + z \frac{z (I_{g}^{p,\eta,m,k}(z))''}{(I_{g}^{p,\eta,m,k}(z))'} - p \right) = p + \sum_{j=1}^{\eta} k_{j} \left( \frac{z((f_{j} \ast g)(z))'}{(f_{j} \ast g)(z)} - p + p - p \sum_{j=1}^{\eta} k_{j} \right) \quad (2.6)$$

Since $f_{j} \in US_{g}^{p}(\delta_{j},\beta_{j},b) \ (1 \leq j \leq \eta)$, we get

$$\Re \left\{ p + \frac{1}{b} \left( 1 + z \frac{z (I_{g}^{p,\eta,m,k}(z))''}{(I_{g}^{p,\eta,m,k}(z))'} - p \right) \right\} \quad (2.7)$$

$$= p + \sum_{j=1}^{\eta} k_{j} \Re \left\{ \frac{1}{b} \left( \frac{z((f_{j} \ast g)(z))'}{(f_{j} \ast g)(z)} - p \right) \right\} + p - p \sum_{j=1}^{\eta} \delta_{j} k_{j}$$

$$> \sum_{j=1}^{\eta} k_{j} \delta_{j} \left| \frac{z((f_{j} \ast g)(z))'}{(f_{j} \ast g)(z)} - p \right| + p + \sum_{j=1}^{\eta} k_{j}(\beta_{j} - p).$$
Since
\[ \sum_{j=1}^{\eta} k_j \delta_j \left\{ \frac{1}{b} \left( \frac{z((f_j * g)(z))'}{(f_j * g)(z)} - p \right) \right\} > 0 \]
because the integral operator \( I_{\eta} t_\eta g(z) \), defined by (1.8), is in the class \( K_{\eta} t_{\eta} g \) with
\[ \tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \]

\section{3 Sufficient Conditions for \( G_{\eta} t_{\eta} m_{\eta} k(z) \)}

\textbf{Theorem 3.1} Let \( \eta \in N, m = (m_1, ..., m_\eta) \in N_0^\eta \) and \( k = (k_1, ..., k_\eta) \in R_+^\eta \). Also let \( b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p \), and \( f_j \in US_{\eta} t_{\eta} g \) for \( 1 \leq j \leq \eta \). If
\[ 0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p, \quad (3.1) \]
then the integral operator \( G_{\eta} t_{\eta} m_{\eta} k(z) \), defined by (1.8), is in the class \( K_{\eta} t_{\eta} g \) where
\[ \tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p). \]

\textbf{Proof}. From the definition (1.8), we observe that \( I_{\eta} t_{\eta} m_{\eta} k(z) \in A_{\eta}(n) \). We can easily see that
\[ (G_{\eta} t_{\eta} m_{\eta} k(z))' = p z^{p-1} \prod_{j=1}^{\eta} \left( \frac{(f_j * g)'(z)}{p z^{p-1}} \right)^{k_j}. \]

Differentiating (3.2) logarithmically and multiplying by \( 'z' \), we obtain
\[ \frac{z (G_{\eta} t_{\eta} m_{\eta} k(z))''}{(G_{\eta} t_{\eta} m_{\eta} k(z))'} = p - 1 + \sum_{j=1}^{\eta} k_j \left( z \left( \frac{(f_j * g)(z))''}{(f_j * g)'(z)} + 1 - p \right) \right) \]
\[ \text{or equivalently} \]
\[ 1 + \frac{z (G_{\eta} t_{\eta} m_{\eta} k(z))''}{(G_{\eta} t_{\eta} m_{\eta} k(z))'} - p = \sum_{j=1}^{\eta} k_j \left( z \left( \frac{(f_j * g)(z))''}{(f_j * g)'(z)} + 1 - p \right) \right) \]
\[ \text{(3.4)} \]

Then, by multiplying (3.4) with \( '1/b' \), we have
\[ \frac{1}{b} \left( 1 + \frac{z (G_{\eta} t_{\eta} m_{\eta} k(z))''}{(G_{\eta} t_{\eta} m_{\eta} k(z))'} - p \right) = \sum_{j=1}^{\eta} k_j \frac{1}{b} \left( z \frac{(f_j * g)(z))''}{(f_j * g)'(z)} + 1 - p \right) \]
\[ \text{(3.5)} \]
or
\[
p + \frac{1}{b} \left( \frac{z \left( \mathcal{G}_{p,n,m,k}(z) \right)''}{\left( \mathcal{G}_{p,n,m,k}(z) \right)'} + 1 - p \right) = p + \sum_{j=1}^{\eta} k_j \frac{1}{b} \left( \frac{z \left( (f_j * g)(z) \right)''}{(f_j * g)'(z)} + 1 - p + p - p \sum_{j=1}^{\eta} k_j \right)
\]
(3.6)

Since \( f_j \in \mathcal{U}K_{\eta}(\delta_j, \beta_j, b) \) (1 \( \leq j \leq \eta \)), we get
\[
\text{Re} \left\{ p + \frac{1}{b} \left( 1 + \frac{z \left( \mathcal{G}_{p,n,m,k}(z) \right)''}{\left( \mathcal{G}_{p,n,m,k}(z) \right)'} - p \right) \right\}
\]
(3.7)
\[
= p + \sum_{j=1}^{\eta} k_j \text{Re} \left\{ \frac{1}{b} \left( \frac{z \left( (f_j * g)(z) \right)''}{(f_j * g)'(z)} + 1 - p \right) \right\} + p - \sum_{j=1}^{\eta} pk_j + p + \sum_{j=1}^{\eta} k_j (\beta_j - p).
\]
\[
> \sum_{j=1}^{\eta} k_j \delta_j \left\{ \frac{1}{b} \left( \frac{z \left( (f_j * g)(z) \right)''}{(f_j * g)'(z)} + 1 - p \right) \right\} + p + \sum_{j=1}^{\eta} k_j (\beta_j - p).
\]

Since
\[
\sum_{j=1}^{\eta} k_j \delta_j \left\{ \frac{1}{b} \left( \frac{z \left( (f_j * g)(z) \right)''}{(f_j * g)'(z)} + 1 - p \right) \right\} > 0
\]
because the integral operator \( \mathcal{G}_{p,n,m,k}(z) \), defined by (1.8), is in the class \( \mathcal{K}_{\eta}(\tau, b) \) with
\[
\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).
\]

4 Corollaries and Consequences

For \( \eta = 1, m_1 = m, k_1 = k \), and \( f_1 = f \), we have

\textbf{Corollary 4.1} Let \( \eta \in N, m \in N_0^\eta \) and \( k \in R_+^\eta \). Also let \( b \in C - [0], \delta \geq 0, 0 \leq \beta < p \), and \( f \in US_{\eta}(\delta, \beta, b) \) for 1 \( \leq j \leq \eta \). If
\[
0 \leq p + k(\beta - p) < p,
\]
then the integral operator \( \mathcal{I}_{\eta}(p,n,m,k)(z) \) is in the class \( \mathcal{K}_{\eta}(\tau, b) \) where
\[
\tau = p + k(\beta - p).
\]

\textbf{Corollary 4.2} Let \( \eta \in N, m \in N_0^\eta \) and \( k \in R_+^\eta \). Also let \( b \in C - [0], \delta \geq 0, 0 \leq \beta < p \), and \( f \in US_{\eta}(\delta, \beta, b) \) for 1 \( \leq j \leq \eta \). If
\[
0 \leq p + k(\beta - p) < p,
\]
then the integral operator \( \mathcal{G}_{\eta}(p,n,m,k)(z) \) is in the class \( \mathcal{K}_{\eta}(\tau, b) \) where
\[
\tau = p + k(\beta - p).
\]
For \((f_j + g)(z) = D^{m,\alpha}_{\lambda,l,p}f_j(z)\), we have

**Corollary 4.3** Let \(\eta \in N, m = (m_1, \ldots, m_\eta) \in N^\eta_0\) and \(k = (k_1, \ldots, k_\eta) \in R^\eta_+\). Also let \(b \in \mathbb{C} \setminus \{0\}, \delta \geq 0, 0 \leq \beta < p\), and \(f_j \in U\mathcal{S}^{m,\alpha}_{\lambda,l,p}(\delta_j, \beta_j, b)\) for \(1 \leq j \leq \eta\). If

\[
0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p, \tag{4.3}
\]

then the integral operator \(I_{p,\eta, m, k}(z)\) is in the class \(\mathcal{K}^{p,n}(\tau, b)\) where

\[
\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).
\]

**Corollary 4.4** Let \(\eta \in N, m = (m_1, \ldots, m_\eta) \in N^\eta_0\) and \(k = (k_1, \ldots, k_\eta) \in R^\eta_+\). Also let \(b \in \mathbb{C} \setminus \{0\}, \delta \geq 0, 0 \leq \beta < p\), and \(U\mathcal{K}^{m,\alpha,\beta,n}_{\lambda,l}(\delta_j, \beta_j, b)\) for \(1 \leq j \leq \eta\). If

\[
0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p, \tag{4.4}
\]

then the integral operator \(G_{p,\eta, m, k}(z)\) is in the class \(\mathcal{K}^{p,n}(\tau, b)\) where

\[
\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).
\]

which are known results obtained by Guney and Bulut [2]. Further, if put \(p = 1\), we have

**Corollary 4.5** Let \(\eta \in N, m = (m_1, \ldots, m_\eta) \in N^\eta_0\) and \(k = (k_1, \ldots, k_\eta) \in R^\eta_+\). Also let \(b \in \mathbb{C} \setminus \{0\}, \delta \geq 0, 0 \leq \beta < 1\), and \(f_j \in U\mathcal{S}^{1}_g(\delta, \beta, b)\) for \(1 \leq j \leq \eta\). If

\[
0 \leq 1 + \sum_{j=1}^{\eta} k_j (\beta_j - 1) < 1, \tag{4.5}
\]

then the integral operator \(I^{1,\alpha,\eta, m, k}_g(z)\) is in the class \(\mathcal{K}^{1}_g(\tau, b)\) where

\[
\tau = 1 + \sum_{j=1}^{\eta} k_j (\beta_j - 1).
\]

**Corollary 4.6** Let \(\eta \in N, m = (m_1, \ldots, m_\eta) \in N^\eta_0\) and \(k = (k_1, \ldots, k_\eta) \in R^\eta_+\). Also let \(b \in \mathbb{C} \setminus \{0\}, \delta \geq 0, 0 \leq \beta < 1\), and \(f_j \in U\mathcal{S}^{1}_g(\delta, \beta, b)\) for \(1 \leq j \leq \eta\). If

\[
0 \leq 1 + \sum_{j=1}^{\eta} k_j (\beta_j - 1) < 1, \tag{4.6}
\]
then the integral operator \( G^{1,n,m,k}_g(z) \) is in the class \( K^1_g(\tau, b) \) where

\[
\tau = 1 + \sum_{j=1}^\eta k_j(\beta_j - 1).
\]

Upon setting \( g(z) = z^p/(1 - z) \), we have

**Corollary 4.7** Let \( \eta \in \mathbb{N}, m = (m_1, \ldots, m_\eta) \in \mathbb{N}_0^\eta \) and \( k = (k_1, \ldots, k_\eta) \in \mathbb{R}^\eta_+ \). Also let \( b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p \), and \( f_j \in \mathcal{US}^p(\delta, \beta, b) \) for \( 1 \leq j \leq \eta \). If

\[
0 \leq p + \sum_{j=1}^\eta k_j(\beta_j - p) < p.
\] (4.7)

then the integral operator \( G^{p,n,m,k}_g(z) \) is in the class \( K^p(\tau, b) \) where

\[
\tau = p + \sum_{j=1}^\eta k_j(\beta_j - p).
\]

**Corollary 4.8** Let \( \eta \in \mathbb{N}, m = (m_1, \ldots, m_\eta) \in \mathbb{N}_0^\eta \) and \( k = (k_1, \ldots, k_\eta) \in \mathbb{R}^\eta_+ \). Also let \( b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p \), and \( f_j \in \mathcal{US}^p(\delta, \beta, b) \) for \( 1 \leq j \leq \eta \). If

\[
0 \leq p + \sum_{j=1}^\eta k_j(\beta_j - p) < p.
\] (4.8)

then the integral operator \( G^{p,n,m,k}_g(z) \) is in the class \( K^p(\tau, b) \) where

\[
\tau = p + \sum_{j=1}^\eta k_j(\beta_j - p).
\]

Upon setting \( g(z) = z^p/(1 - z) \) and \( \delta = 0 \), we have

**Corollary 4.9** Let \( \eta \in \mathbb{N}, m = (m_1, \ldots, m_\eta) \in \mathbb{N}_0^\eta \) and \( k = (k_1, \ldots, k_\eta) \in \mathbb{R}^\eta_+ \). Also let \( b \in \mathbb{C} - \{0\}, 0 \leq \beta < p \), and \( f_j \in \mathcal{US}^p(0, \beta, b) \) for \( 1 \leq j \leq \eta \). If

\[
0 \leq p + \sum_{j=1}^\eta k_j(\beta_j - p) < p.
\] (4.9)

then the integral operator \( G^{p,n,m,k}_g(z) \) is in the class \( K^p(\tau, b) \) where

\[
\tau = p + \sum_{j=1}^\eta k_j(\beta_j - p).
\]
Corollary 4.10 Let \( \eta \in \mathbb{N}, m = (m_1, ..., m_\eta) \in \mathbb{N}_0^\eta \) and \( k = (k_1, ..., k_\eta) \in \mathbb{R}_+^\eta \). Also let \( b \in \mathbb{C} - \{0\}, \delta \geq 0, 0 \leq \beta < p \), and \( f_j \in \mathcal{US}^p(0, \beta, b) \) for \( 1 \leq j \leq \eta \). If

\[
0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p,
\]

then the integral operator \( G^{p, \eta, m, k}(z) \) is in the class \( \mathcal{K}^p(\tau, b) \) where

\[
\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).
\]

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