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Remarks concerning the pexiderized Gołąb–Schinzel functional equation

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Abstract: This paper is devoted to proof of theorem concerning solutions of the pexiderized Gołąb–Schinzel functional equation. We provide explicite formulas expressing solutions of the equation. Our considerations refer to the paper [6].

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In the paper we consider the pexiderized Gołąb–Schinzel functional equation, i.e. the equation

$$f(x + g(x)y) = h(x)k(y) \tag{1}$$

in the class of unknown functions $f, g, h, k : X \to \mathbb{K}$, where X is a linear space over a commutative field $\mathbb{K}$. This equation generalizes one of the Pexider equations, i.e.

$$f(x + y) = g(x)h(y),$$

which is very well-known for over hundred years (see [7]), as well as the Gołąb–Schinzel equation

$$f(x + f(x)y) = f(x)f(y),$$

which appeared in 1959 in [4] and has been extensively studied by many authors (for more information see a survey paper [1]).

In 1966 E. Vincze introduced equation (1) in [8]. Next papers concerning it have been published over forty years later (see [2], [6]).

The principal aim of the paper is to prove the theorem, which characterizes general solutions of the equation (1) combined with a partially pexiderized Gołąb–Schinzel equation, i.e. the equation

$$f(x + g(x)y) = f(x)f(y). \tag{2}$$

Our main result is:

Theorem 1. (cf. [6, Theorem 1]) *Let X be a linear space over a commutative field $\mathbb{K}$. Functions $f, g, h, k : X \to \mathbb{K}$ satisfy* (1) *iff they have one of the following forms:*

(i) $\begin{cases} f = 0, \\ h = 0, \\ g, k \text{ are arbitrary} \end{cases}$ *or* $\begin{cases} f = 0, \\ k = 0, \\ g, h \text{ are arbitrary;} \end{cases}$

(ii) *there are $a, b \in \mathbb{K} \setminus \{0\}$ such that* $\begin{cases} f = ab, \\ g \text{ is arbitrary}, \\ h = a, \\ k = b; \end{cases}$

(iii) *there is a $b \in \mathbb{K} \setminus \{0\}$ such that* $\begin{cases} f = bh, \\ g = 0, \\ h \text{ is arbitrary nonconstant}, \\ k = b; \end{cases}$

(iv) *there are $a, b, c \in \mathbb{K} \setminus \{0\}$ and functions $F, G : X \to \mathbb{K}$ with $F \neq 1$ and $F(0) = G(0) = 1$, such that F and G satisfy the equation* (2) *and*
$\begin{cases} f = abF, \\ g = cG, \\ h = aF, \\ k(x) = bF(cx) \text{ for } x \in X; \end{cases}$

(v) *there are $x_0 \in X \setminus \{0\}$, $a, b \in \mathbb{K} \setminus \{0\}$ and functions $F, G : X \to \mathbb{K}$ with $F(0) = G(0) = 1$, $F(-x_0) = G(-x_0) = 0$, such that F and G satisfy the equation* (2) *and* $\begin{cases} f(x) = abF(x - x_0) & \text{for } x \in X, \\ g(x) = g(x_0)G(x - x_0) & \text{for } x \in X, \\ h(x) = aF(x - x_0) & \text{for } x \in X, \\ k(x) = bF(g(x_0)x) & \text{for } x \in X. \end{cases}$

Proof. By [6, Theorem 1 (i)–(iv)] conditions (i)–(iv) of the theorem holds. Now we have to prove (v). According to [6, Theorem 1(v)] there are $x_0 \in X \setminus \{0\}$, $a, b \in \mathbb{K} \setminus \{0\}$ and a function $f_0 : X \to \mathbb{K}$ with

$$f_0(x_0) = 1, \quad f_0(0) = g(0) = 0, \tag{3}$$

such that f_0 and g satisfy the equation

$$f_0(x + g(x)y) = f_0(x) f_0(x_0 + g(x_0)y) \text{ for every } x, y \in X \tag{4}$$

and

$$\begin{cases} f = abf_0, \\ h = af_0, \\ k(x) = bf_0(x_0 + g(x_0)x) \text{ for } x \in X. \end{cases} \tag{5}$$

First consider the case, when $g(x_0) = 0$. Then equation (4) has the following form:

$$f_0(x + g(x)y) = f_0(x).$$

Suppose that $g(y_0) \neq 0$ for some $y_0 \in X$. Then, for every $z \in X$, there exists a $y \in X$ such that $z = y_0 + g(y_0)y$ and hence, by (4),

$$f_0(z) = f_0(y_0 + g(y_0)y) = f_0(y_0) \text{ for every } z \in X.$$

It means that f_0 is constant, what contradicts (3). So, $g = 0$ and f_0 is arbitrary. Hence, by (5), $f = abf_0$, $g = 0$, $h = af_0$ and $k = b$ with an arbitrary function f_0. Thus $f = bh$, $g = 0$, h is arbitrary and $k = b$ and consequently functions f, g, h, k have the same form as in condition (iii).

Now we consider the case, when $g(x_0) \neq 0$. Define functions $F, G : X \to \mathbb{K}$ as follows:

$$F(x) = f_0(x + x_0) \text{ for } x \in X,$$
$$G(x) = \tfrac{g(x+x_0)}{g(x_0)} \text{ for } x \in X.$$

Clearly $F(0) = G(0) = 1$ and $F(-x_0) = G(-x_0) = 0$. Moreover, by (4), for every $x, y \in X$ we have:

$$\begin{aligned} F(x + G(x)y) &= F\left(x + \tfrac{g(x+x_0)}{g(x_0)}y\right) = f_0\left(x + x_0 + g(x + x_0)\tfrac{y}{g(x_0)}\right) \\ &= f_0(x + x_0)f_0\left(x_0 + g(x_0)\tfrac{y}{g(x_0)}\right) = F(x)F(y). \end{aligned}$$

Hence functions F, G satisfy (2), what ends the proof of condition (v). □

Theorem 1 shows that the pexiderized Gołąb–Schinzel equation is tightly connected with the equation (2). The equation (2) has been considered by J. Chudziak [3] in the class of real functions f, g, where g is continuous, or by the author of [5] in the class of continuous on rays functions $f, g : X \to \mathbb{R}$ (where X is a real linear space).

Using Theorem 1 and the result of J. Chudziak [3, Theorem 1], we obtain the following corollary.

Corollary 1. *Functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ satisfy (1) and g is continuous if and only if they have one of the following forms:*

(i) $\begin{cases} f = 0, \\ g \textit{ is arbitrary continuous}, \\ h = 0, \\ k \textit{ is arbitrary}, \end{cases}$ *or* $\begin{cases} f = 0, \\ g \textit{ is arbitrary continuous}, \\ h \textit{ is arbitrary}, \\ k = 0; \end{cases}$

(ii) *there are $a, b \in \mathbb{R} \setminus \{0\}$ such that* $\begin{cases} f = ab, \\ g \textit{ is arbitrary continuous}, \\ h = a, \\ k = b; \end{cases}$

(iii) *there is a $b \in \mathbb{R} \setminus \{0\}$ such that* $\begin{cases} f = bh, \\ g = 0, \\ h \textit{ is arbitrary nonconstant}, \\ k = b; \end{cases}$

(iv) *there are* $a, b, c \in \mathbb{R} \setminus \{0\}$ *such that* $\begin{cases} f = abF, \\ g = cG, \\ h = aF, \\ k(x) = bF(cx) \ \ \textit{for} \ \ x \in \mathbb{R}, \end{cases}$

where $F, G : \mathbb{R} \to \mathbb{R}$ *are defined by one of the following three formulas:*

- $G = 1$ *and* F *is an exponential function;*
- *there are a nonconstant multiplicative function* $\phi : \mathbb{R} \to \mathbb{R}$ *and* $d \in \mathbb{R} \setminus \{0\}$ *such that*

$$\begin{cases} G(x) = dx + 1 & \textit{for} \ \ x \in \mathbb{R}, \\ F(x) = \phi(dx + 1) & \textit{for} \ \ x \in \mathbb{R}; \end{cases} \tag{6}$$

- *there are a nonconstant multiplicative function* $\phi : [0, \infty) \to [0, \infty)$ *and* $d \in \mathbb{R} \setminus \{0\}$ *such that*

$$\begin{cases} G(x) = \max\{dx + 1, 0\} & \textit{for} \ \ x \in \mathbb{R}, \\ F(x) = \phi(\max\{dx + 1, 0\}) & \textit{for} \ \ x \in \mathbb{R}; \end{cases} \tag{7}$$

(v) *there are* $a, b, c, d \in \mathbb{R} \setminus \{0\}$ *such that either*

$$\begin{cases} f(x) = ab\phi(dx) & \textit{for} \ \ x \in \mathbb{R}, \\ g(x) = cdx & \textit{for} \ \ x \in \mathbb{R}, \\ h(x) = a\phi(dx) & \textit{for} \ \ x \in \mathbb{R}, \\ k(x) = b\phi(cdx + 1) & \textit{for} \ \ x \in \mathbb{R}, \end{cases} \tag{8}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ *is a nonconstant multiplicative function, or*

$$\begin{cases} f(x) = ab\phi(\max\{dx, 0\}) & \textit{for} \ \ x \in \mathbb{R}, \\ g(x) = c\max\{dx, 0\} & \textit{for} \ \ x \in \mathbb{R}, \\ h(x) = a\phi(\max\{dx, 0\}) & \textit{for} \ \ x \in \mathbb{R}, \\ k(x) = b\phi(\max\{cdx + 1, 0\}) & \textit{for} \ \ x \in \mathbb{R}, \end{cases} \tag{9}$$

where $\phi : [0, \infty) \to [0, \infty)$ *is a nonconstant multiplicative function.*

In the same way, using Theorem 1 and [5, Theorem 1], the following corollary can be derived.

Corollary 2. *Let* X *be a real linear space. Functions* $f, g, h, k : X \to \mathbb{R}$ *satisfy* (1) *and* f, g *are continuous on rays if and only if they have one of the following forms:*

(i) $\begin{cases} f = 0, \\ g \textit{ is arbitrary continuous on rays}, \\ h = 0, \\ k \textit{ is arbitrary}, \end{cases}$ *or* $\begin{cases} f = 0, \\ g \textit{ is arbitrary continuous on rays}, \\ h \textit{ is arbitrary}, \\ k = 0; \end{cases}$

(ii) *there are some* $a, b \in \mathbb{R} \setminus \{0\}$ *such that* $\begin{cases} f = ab, \\ g \textit{ is arbitrary continuous on rays}, \\ h = a, \\ k = b; \end{cases}$

(iii) *there is a* $b \in \mathbb{R} \setminus \{0\}$ *such that* $\begin{cases} f = bh, \\ g = 0, \\ h \text{ is arbitrary nonconstant continuous on rays}, \\ k = b; \end{cases}$

(iv) *there are a nontrivial linear functional* $L : X \to \mathbb{R}$, $a, b, c \in \mathbb{R} \setminus \{0\}$ *and* $r > 0$ *such that* $\begin{cases} f = abF, \\ g = cG, \\ h = aF, \\ k(x) = bF(cx) \ \text{for } x \in X, \end{cases}$
where F *and* G *are defined by one of the following four formulas:*

- $G = 1$ *and* $F = \exp L$;
- $\begin{cases} G(x) = L(x) + 1 & \text{for } x \in X, \\ F(x) = |L(x) + 1|^r & \text{for } x \in X; \end{cases}$
- $\begin{cases} G(x) = L(x) + 1 & \text{for } x \in X, \\ F(x) = |L(x) + 1|^r \operatorname{sgn}(L(x) + 1) & \text{for } x \in X; \end{cases}$
- $\begin{cases} G(x) = \max\{L(x) + 1, 0\} & \text{for } x \in X, \\ F(x) = (\max\{L(x) + 1, 0\})^r & \text{for } x \in X; \end{cases}$

(v) *there are a nontrivial linear functional* $L : X \to \mathbb{R}$, $a, b, c \in \mathbb{R} \setminus \{0\}$ *and* $r > 0$ *such that either* $\begin{cases} f = ab(\phi \circ L), \\ g = cL, \\ h = a(\phi \circ L), \\ k(x) = b\phi(1 + cL(x)) \ \text{for } x \in X, \end{cases}$
where $\phi : \mathbb{R} \to \mathbb{R}$ *has one of the following two forms:*

$$\phi(\alpha) = |\alpha|^r \ \text{for } \alpha \in \mathbb{R} \quad \text{or} \quad \phi(\alpha) = |\alpha|^r \operatorname{sgn} \alpha \ \text{for } \alpha \in \mathbb{R},$$

or $\begin{cases} f(x) = ab(\max\{L(x), 0\})^r & \text{for } x \in X, \\ g(x) = c \max\{L(x), 0\} & \text{for } x \in X, \\ h(x) = a(\max\{L(x), 0\})^r & \text{for } x \in X, \\ k(x) = b(\max\{cL(x) + 1, 0\})^r & \text{for } x \in X. \end{cases}$

At the end of the paper let us mention that equation (1) has been treated in [6] in the class of real continuous functions f, g, h, k (see [6, Corollary 1]), but the proof given there is not correct, because [6, Proposition 1] does not hold (to see this it is enough to choose functions $f(x) = g(x) = \max\{x, 0\}$ for $x \in \mathbb{R}$). Consequently, [6, Theorem 2] and [6, Corollary 1] were not stated thoroughly, because their proofs base on [6, Proposition 1].

References

[1] Brzdęk J.: The Gołąb–Schinzel equation and its generalizations. *Aequationes Math.* **70**, 14–24 (2005).

[2] Charifi A., Bouikhalene B. and Kabbaj S.: On solutions of Pexiderizations of the Gołąb–Schinzel Functional Equation. *Inequality Theory and Applications, Nov. Sc. Publ.* **6**, 25–36 (2010).

[3] Chudziak J.: Semigroup–Valued Solutions of the Gołąb–Schinzel Functional Equation. *Abh. Math. Sem. Univ. Hamburg* **76**, 91–98 (2006).

[4] Gołąb S. and Schinzel A.: Sur l'équation fonctionnelle $f(x + f(x)y) = f(x)f(y)$. *Publ. Math. Debrecen* **6**, 113-125 (1959).

[5] Jabłońska E.: Continuous on rays solutions of an equation of the Gołąb–Schinzel type. *J. Math. Anal. Appl.* **375**, 223–229 (2011).

[6] Jabłońska E.: The pexiderized Gołąb–Schinzel functional equation. *J. Math. Anal. Appl.* **381**, 565–572 (2011).

[7] Pexider H.W.: Notiz über Funktionaltheoreme. *Monatsh. Math. Phys.* **14**, 293-301 (1903).

[8] Vincze E.: Über die Lösung der Funktionalgleichung $f(y + xg(y)) = L(h(x), k(y))$. *Ann. Polon. Math.* **18**, 115–119 (1966).

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