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The reticulation of residuated lattices induced by fuzzy prime spectrum

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Abstract: In this paper, we use the fuzzy prime spectrum to define the reticulation $(L(A), \lambda)$ of a residuated lattice A. We obtain some related results. In particular, we show that the lattices of fuzzy filters of a residuated lattice A and $L(A)$ are isomorphic and the fuzzy prime spectrum of A and $L(A)$ are homomorphic topological space.

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1 Introduction

M. Ward and R.P. Dilworth [12] introduced the concept of residuated lattices as generalization of ideal lattices of rings. These algebras have been widely studied (See [1], [2] and [6]).
The reticulation was first defined by simmons([10]) for commutative ring and L. Leustean made this construction for BL-algebras ([7]). C. Mureson defined the reticulations for residuated lattices ([8]). The reticulation of an algebra A is a pair $(L(A), \lambda)$ consisting of a bounded distributive lattice $L(A)$ and a surjective $\lambda : A \to L(A)$([8]). Hence we can transfer many properties between A and $L(A)$.
The concept of fuzzy sets were introduced by Zadeh in 1965 ([13]). This concept was applied to residuated lattices and proposed the notions of fuzzy filters and prime fuzzy filters in a residuated lattice ([3], [4] and [14]). We defined and studied fuzzy prime spectrum of a residuated lattice in ([5]).
In this paper, we use fuzzy prime spectrum to define the congruence relation $\cong$ on a residuated lattice A. Then we will show that $A/\cong$ is a bounded distributive lattice and $(A/\cong, \pi)$ is a reticulation of A. We will investigate some related results. Also, we obtain the relation between the reticulation of a residuated lattice induced by fuzzy prime spectrum and the reticulation of a residuated lattice which is defined in ([8]).

2 Preliminaries

We recall some definitions and theorems which will be needed in this paper.

Definition 2.1. ([1], [11]) A *residuated lattice* is an algebraic structure $(A, \wedge, \vee, \rightarrow, *, 0, 1)$ such that
(1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1,
(2) $(A, *, 1)$ is a commutative monoid where 1 is a unit element,
(3) $x * y \leq z$ iff $x \leq y \rightarrow z$, for all $x, y, z \in A$.

In the rest of this paper, we denote the residuated lattice $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ by A.

Proposition 2.2. ([6], [11]) Let A be a residuated lattice. Then we have the following properties: for all $x, y, z \in A$,
(1) $x \leq y$ if and only if $x \rightarrow y = 1$,
(2) $x * y \leq x \wedge y \leq x, y$,
(3) $x * (y \vee z) = (x * y) \vee (x * z)$.

Definition 2.3. ([6], [11]) Let F be a non-empty subset of a residuated lattice A. F is called a *filter* if
(1) $1 \in F$,
(2) if $x, x \rightarrow y \in F$, then $y \in F$, for all $x, y \in A$.
F is called *proper*, if $F \neq A$.

Theorem 2.4. ([6], [11]) A non-empty subset F of a residuated lattice A is a filter if and only if
(1) $x, y \in F$ implies $x * y \in F$,
(2) if $x \leq y$ and $x \in F$, then $y \in F$.

Definition 2.5. ([6]) Let X be a subset of a residuated lattice A. The smallest filter of A which contains X is said to be the *filter generated* by X and will be denoted by $< X >$.

Proposition 2.6. ([6]) Let X be a non-empty subset of a residuated lattice A. Then $< X >= \{a \in A : a \geq x_1 * \ldots * x_n \text{ for some } x_1, \ldots, x_n \in X\}$.

Definition 2.7. ([6], [11]) A proper filter F of a residuated lattice A is called *prime filter*, if for all $x, y \in A$, $x \vee y \in A$, implies $x \in A$ or $y \in A$.

Proposition 2.8. ([9]) (The prime filter theorem) Let A be a residuated lattice, F be a filter of A and $a \in A \backslash F$. Then there exists a prime filter of A that includes F and does not contain a.

Definition 2.9. ([13]) Let X be a non-empty subset. A *fuzzy set* in X is a mapping

$\mu : X \longrightarrow [0,1]$. For $t \in [0,1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a *level subset* of μ. We call that μ is *proper*, if it has more two distinct values.

Definition 2.10. Let X, Y be non-empty sets and $f : X \to Y$ be a function. Let μ be a fuzzy set in X and ν be a fuzzy set in Y. Then $f(\mu)$ is a fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) : x \in f^{-1}(y)\} & if \quad f^{-1}(y) \neq \emptyset \\ 0 & if \quad f^{-1}(y) = \emptyset \end{cases}$$

for all $y \in Y$ and $f^{-1}(\nu)$ is a fuzzy set in X defined by $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$.

Definition 2.11. Let X be a lattice. A fuzzy set μ is called a *fuzzy lattice filter* in X if it satisfies: for all $x, y \in X$,
(1) $\mu(x) \leq \mu(1)$,
(2) $min\{\mu(x), \mu(y)\} \leq \mu(x \wedge y)$.
The set of all fuzzy lattice filter in X is denoted by $\mathcal{FL}(X)$.

Definition 2.12. ([3], [14]) Let A be a residuated lattice. A fuzzy set μ is called a *fuzzy filter* in A if it satisfies: for all $x, y \in A$,
$(fF1)$ $\mu(x) \leq \mu(1)$,
$(fF4)$ $min\{\mu(x), \mu(x \to y)\} \leq \mu(y)$.
The set of all fuzzy filter in A is denoted by $\mathcal{F}(A)$.

Theorem 2.13. ([3], [14]) Let A be a residuated lattice. A fuzzy set μ in A is a fuzzy filter if and only if it satisfies: for all $x, y \in A$,
$(fF1)$ $x \leq y$ imply $\mu(x) \leq \mu(y)$,
$(fF2)$ $min\{\mu(x), \mu(y)\} \leq \mu(x * y)$.

Proposition 2.14. ([3]) Let μ be a fuzzy filter of A. If $\mu(x \to y) = \mu(1)$, then $\mu(x) \leq \mu(y)$, for any $x, y \in A$.

Definition 2.15. Let μ be a fuzzy set in a residuated lattice A. The smallest fuzzy filter in A which contains μ is said to be the *fuzzy filter generated* by μ and will be denoted by $< \mu >$.

Proposition 2.16. Let μ be a fuzzy set of a residuated lattice A. Then $< \mu > (x) = \sup\{\min\{\mu(a_1), \ldots, \mu(a_n)\} : x \geq a_1 * \ldots * a_n$ for some $a_1, \ldots, a_n \in X\}$, for all $x \in A$.

Definition 2.17. ([5]) Let μ be a proper fuzzy filter in a residuated lattice A. μ is called a *fuzzy prime filter* if $\mu(x \vee y) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in A$.

Theorem 2.18. ([5]) A proper subset P of a residuated lattice A is a prime filter of A if and only if χ_P is a fuzzy prime filter in A.

Theorem 2.19. ([5]) Let A and A' be residuated lattices and $f : A \to A'$ be an epimorphism. If μ is a fuzzy prime filter in A which is constant on $\ker(f)$, then $f(\mu)$ is a fuzzy prime filter in A'.

Theorem 2.20. ([5]) Let A and A' be residuated lattices and $f : A \to A'$ be a homomorphism. If ν is a fuzzy prime filter in A', then $f^{-1}(\nu)$ is a fuzzy prime filter in A.

Notation: ([5]) We shall denote the set of all fuzzy prime filter μ in a residuated lattice A such that $\mu(1) = 1$ by $Fspec(A)$. For each fuzzy set ν in A, define $\mathcal{C}(\nu) = \{\mu \in Fspec(A) : \nu \leq \mu\}$. Let $\mu = \chi_{\{a\}}$ for $a \in A$. We shall denote $\mathcal{C}(\mu)$ by $\mathcal{C}(a)$ for all $a \in A$. Thus $\mathcal{C}(a) = \{\mu \in Fspec(A) : \mu(a) = 1\}$.

Proposition 2.21 ([5]) Let μ, ν be fuzzy sets in a residuated lattice A and $a, b \in A$. Then
(1) $\mu \leq \nu$ imply $\mathcal{C}(\nu) \subseteq \mathcal{C}(\mu) \subseteq Fspec(A)$.
(2) $\mathcal{C}(\bigcup\limits_{i\in I} \nu_i) = \bigcap\limits_{i\in I} \mathcal{C}(\nu_i)$.
(3) $\mathcal{C}(\mu) \cup \mathcal{C}(\nu) \subseteq \mathcal{C}(<\mu> \cap <\nu>)$.
(4) $\mathcal{C}(a \wedge b) = \mathcal{C}(a) \cup \mathcal{C}(b)$,
(5) $\mathcal{C}(\chi_A) = \bigcap\limits_{a\in A} \mathcal{C}(a)$.

Theorem 2.22.([5]) Let $\mathcal{V}(a) = Fspec(A)\backslash\mathcal{C}(a)$ and $\mathcal{B} = \{\mathcal{V}(a) : a \in A\}$. Then $\mathcal{B}$ is a base for a topology on $Fspec(A)$. The topological space $Fspec(A)$ is called *fuzzy spectrum* of A.

3 The reticulation of residuated lattices

Definition 3.1. Let A be a residuated lattice. Define

$$a \cong b \qquad \text{if and only if} \qquad \mathcal{C}(a) = \mathcal{C}(b),$$

for all $a, b \in A$. Hence $a \cong b$ iff for any $\mu \in Fspec(A)$, ($\mu(a) = 1$ iff $\mu(b) = 1$).

Theorem 3.2. The relation $\cong$ is a congruence relation on a residuated lattice A with respect to $*$, $\wedge$ and $\vee$.

Proof: It is clear that $\cong$ is an equivalence relation on A. Suppose that $a \cong b$ and $c \cong d$ where $a, b, c, d \in A$. We will show that $a*c \cong b*d$, $a\wedge c \cong b\wedge d$ and $a\vee c \cong b\vee d$.
(1) Let $\mu \in \mathcal{C}(a * c)$. So $\mu(a * c) = 1$. By Proposition 2.2 part (2) and Theorem 2.13, we have $1 = \mu(a * c) \leq \mu(a), \mu(c)$. We get that $\mu(a) = \mu(c) = 1$. By assumption, $\mu(b) = \mu(d) = 1$. Since $b * d \leq b * d$, then $d \leq b \to (b * d)$ by Definition 2.1 part (3). We obtain that $1 = \mu(d) \leq \mu(b \to b * d)$ by Theorem 2.13. Since μ is a fuzzy filter in A, we have $1 = \min\{\mu(b), \mu(b \to b * d)\} \leq \mu(b * d)$. Then $\mu(b * d) = 1$, that is $\mu \in \mathcal{C}(b*d)$. Hence $\mathcal{C}(a*c) \subseteq \mathcal{C}(b*d)$. Similarly, we can show that $\mathcal{C}(b*d) \subseteq \mathcal{C}(a*c)$. Therefore $a * c \cong b * d$.

(2) Let $a \wedge c \cong b \wedge d$ and $\mu \in \mathcal{C}(a \wedge c)$. Thus $\mu(a \wedge c) = 1$. Since $a \wedge c \leq a, c$, then $1 = \mu(a \wedge c) \leq \mu(a), \mu(c)$ by Theorem 2.13. By assumption $\mu(b) = \mu(d) = 1$. Since μ is a fuzzy filter in A and $b * d \leq b \wedge d$, then $1 = \min\{\mu(b), \mu(d)\} \leq \mu(b * d) \leq \mu(b \wedge d)$ by Theorem 2.13. Hence $\mu(b \wedge d) = 1$ and then $\mathcal{C}(a \wedge c) \subseteq \mathcal{C}(b \wedge d)$. Similarly, we can show that $\mathcal{C}(b \wedge d) \subseteq \mathcal{C}(a \wedge c)$. Therefor $a \wedge c \cong b \wedge d$.
(3) Let $a \vee c \cong b \vee d$ and $\mu \in \mathcal{C}(a \vee b)$. Then $\mu(a \vee b) = 1$. Since μ is a fuzzy prime filter in A, we have $\mu(a) = 1$ or $\mu(b) = 1$. By assumption $\mu(c) = 1$ or $\mu(d) = 1$. Hence $\mu(c \vee d) = \max\{\mu(c), \mu(d)\} = 1$. We obtain that $\mu \in \mathcal{C}(c \vee d)$ and then $\mathcal{C}(a \vee b) \subseteq \mathcal{C}(c \vee d)$. Similarly, we can prove that $\mathcal{C}(c \vee d) \subseteq \mathcal{C}(a \vee b)$. Hence $a \vee c \cong b \vee d$. ■

Notation: Let $\cong$ be a the congruence relation on residuated lattice A which is defined in Definition 3.1. For all $a \in A$, the equivalence class of a is denoted by $[a]$, that is $[a] = \{b \in A : a \cong b\}$. The set of all equivalence classes is denoted by $A/\cong$.

Theorem 3.3. The algebra $(A/\cong, \wedge, \vee, [0], [1])$ is a bounded lattice, where

$$[a] \vee [b] = [a \vee b] \text{ and } [a] \wedge [b] = [a \wedge b]$$

for all $a, b \in A$.

Proof: By Theorem 3.2, the operation $\wedge$ and $\vee$ are well defined. The rest of the proof is routine. ■

Example 3.4. Consider the residuated lattice A with the universe $\{0, a, b, c, d, 1\}$. Lattice ordering is such that $0 < a, b < c < 1$, $0 < b < d < 1$ but $\{a, b\}$ and $\{c, d\}$ are incomparable. The operations of $*$ and $\rightarrow$ are given by the tables below :

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Consider $0 \leq \nu_1(0) = \nu_1(a) = \nu_1(b) = \nu_1(c) < \nu_1(d) = \nu_1(1) = 1$ and $0 \leq \nu_2(0) = \nu_2(b) = \nu_2(d) < \nu_2(c) = \nu_2(c) = \nu_2(1) = 1$. Then $Fspec(A) = \{\nu_1, \nu_2\}$. We obtain that $[0] = [b]$, $[a] = [c]$. Therefore $A/\cong = \{[0], [a], [d], [1]\}$ where $[0] < [a], [d] < 1$ but $\{[a], [d]\}$ are incomparable.

Lemma 3.5. Let A be a residuated lattice and $a, b \in A$. Then
(i) $[a] \leq [b]$ if and only if $\mathcal{C}(b) \subseteq \mathcal{C}(a)$,
(ii) if $a \leq b$, then $[a] \leq [b]$,
(iii) $[a \wedge b] = [a * b]$.

Proof: (i) By Theorem 3.3 and Proposition 2.21 parts (1) and (4), we have $[a] \leq [b]$ iff $[a] \wedge [b] = [a]$ iff $[a \wedge b] = [a]$ iff $\mathcal{C}(a) = \mathcal{C}(a \wedge b) = \mathcal{C}(a) \cup \mathcal{C}(b)$ iff $\mathcal{C}(b) \subseteq \mathcal{C}(a)$.
(ii) If $a \leq b$, then $\mathcal{C}(a) \subseteq \mathcal{C}(b)$. We obtain that $[a] \leq [b]$ by (i).
(iii) We will show that $\mathcal{C}(a * b) = \mathcal{C}(a \wedge b)$. Let $\mu \in \mathcal{C}(a * b)$. Then $\mu(a * b) = 1$. By Proposition 2.2 part (2) and Theorem 2.13, $\mu(a * b) \leq \mu(a \wedge b)$. We get that $\mu(a \wedge b) = 1$ and then $\mu \in \mathcal{C}(a \wedge b)$. Hence $\mathcal{C}(a * b) \subseteq \mathcal{C}(a \wedge b)$.
Conversely, let $\mu \in \mathcal{C}(a \wedge b)$. Then $\mu(a \wedge b) = 1$. Since $a \wedge b \leq a, b$, then $\mu(a) = \mu(b) = 1$ by Theorem 2.13. Since $b \leq a \to (a * b)$ and μ is a fuzzy filter in A,

$$1 = \min\{\mu(a), \mu(b)\} \leq \min\{\mu(a), \mu(a \to (a * b))\} \leq \mu(a * b).$$

Hence $\mu(a * b) = 1$ and then $\mu \in \mathcal{C}(a * b)$. We get that $\mathcal{C}(a \wedge b) \subseteq \mathcal{C}(a * b)$. Therefor $[a \wedge b] = [a * b]$. ■

Theorem 3.6. The bounded lattice $(A/\cong, \wedge, \vee, [0], [1])$ is distributive.

Proof: Let $a, b, c \in A$. By Lemma 3.5 and Proposition 2.2 part (3),

$$\begin{aligned} [a] \wedge ([b] \vee [c]) &= [a \wedge (b \vee c)] = [a * (b \vee c)] \\ &= [(a * b) \vee (a * c)] = [a * b] \vee [a * c] \\ &= [a \wedge b] \vee [a \wedge c] = ([a] \wedge [b]) \vee ([a] \wedge [c]). \blacksquare \end{aligned}$$

Definition 3.7. Let A be a residuated lattice and $\pi : A \to A/\cong$ be that canonical surjective map defined by $\pi(a) = [a]$. Then $(A/\cong, \pi)$ is called *the reticulation* of residuated lattice induced by fuzzy filters.

Lemma 3.8. Let A_1 and A_2 be residuated lattices and $f : A_1 \to A_2$ be a homomorphism of residuated lattices. Then $\mathcal{C}(a) = \mathcal{C}(b)$ implies $\mathcal{C}(f(a)) = \mathcal{C}(f(b))$, for any $a, b \in A_1$.

Proof: Suppose that $\mathcal{C}(a) = \mathcal{C}(b)$ where $a, b \in A_1$ and $\nu \in \mathcal{C}(f(a))$. Then $\nu \in Fspec(A_2)$ and $\nu(f(a)) = 1$. By Theorem 2.20, we have $f^{-1}(\nu) \in Fspec(A_1)$ and $f^{-1}(\nu)(a) = \nu(f(a)) = 1$. Thus $f^{-1}(\nu) \in \mathcal{C}(a) = \mathcal{C}(b)$. We get that $\nu(f(b)) = f^{-1}(\nu)(b) = 1$ and then $\nu \in \mathcal{C}(f(b))$. Hence $\mathcal{C}(f(a)) \subseteq \mathcal{C}(f(b))$. Similarly, we can show that $\mathcal{C}(f(b)) \subseteq \mathcal{C}(f(a))$.■

In the following theorem, we will define a functor from the category of residuated lattices to the category of bounded distributive lattices.

Theorem 3.9. Let A_1 and A_2 be residuated lattices and $f : A_1 \to A_2$ be a homomorphism of residuated lattices. Then $\bar{f} : A_1/\cong \to A_2/\cong$ is defined by $\bar{f}([a]) = [f(a)]$ is a homomorphism of lattices.

Proof: Let $[a] = [b]$. By Lemma 3.5 part (i), we obtain that $\mathcal{C}(a) = \mathcal{C}(b)$. By Lemma 3.8, $\mathcal{C}(f(a)) = \mathcal{C}(f(b))$. We have $[f(a)] = [f(b)]$ by Lemma 3.5 part (i). So $\bar{f}$ is well defined. Now, Let $a, b \in A_1$. Since f is a homomorphism of residuated lattices, then

$$\bar{f}([a] \wedge [b]) = \bar{f}([a \wedge b]) = [f(a \wedge b)] = [f(a)] \wedge [f(b)] = \bar{f}([a]) \wedge \bar{f}([b]).$$

Similarly, we can show that $\bar{f}([a] \vee [b]) = \bar{f}([a]) \vee \bar{f}([b])$. Also, $\bar{f}([0]) = [f(0)] = [0]$ and $\bar{f}([1]) = [f(1)] = [1]$. Hence $\bar{f}$ is a homomorphism of lattices.■

Lemma 3.10. Let μ be a fuzzy filter in a residuated lattice A and $a, b \in A$ such that $[a] = [b]$. Then $\mu(a) = \mu(b)$.

Proof: Suppose that μ is a fuzzy filter in A such that $\mu(a) \neq \mu(b)$. Then $\mu(a) < \mu(b)$ or $\mu(b) < \mu(a)$. Let $\mu(a) < \mu(b)$. Put $F = \{x \in A : \mu(x) \geq \mu(b)\}$, i.e. $F = \mu_{\mu(b)}$. Hence F is a filter of A such that $a \notin F$. Define $J =< F \cup \{b\} >$. Then J is a filter of A. We shall show that $a \notin J$. Suppose that $a \in J$. By Proposition 2.6, there exist $y_1, ..., y_n \in F \cup \{b\}$ such that $y_1 * ... * y_n \leq a$. If $y_i = b$ for some $1 \leq i \leq n$, then $y_1 * ... * y_{i-1} * y_{i+1} ... * y_n * b \leq a$. Hence $y_1 * ... * y_{i-1} * y_{i+1} ... * y_n \leq b \to a$. Since F is a filter, we have $b \to a \in F$, that is $\mu(b \to a) \geq \mu(b)$. So $\mu(b) = \min\{\mu(b), \mu(b \to a)\} \leq \mu(a)$ which is a contradiction. Now, suppose that $y_i \in F$ for all $1 \leq i \leq n$. Thus $y_1 * ... * y_n \in F$. We get that $a \in F$ which is a contradiction. Hence $a \notin J$ and J is a proper filter. By Proposition 2.8, there exists a prime filter P such that $J \subseteq P$ and $a \notin P$. By Theorem 2.18, χ_P is a fuzzy prime filter in A such that $\chi_P(b) = 1$ and $\chi_P(a) \neq 1$. We obtain that $\chi_P \in \mathcal{C}(b)$ but $\chi_P \notin \mathcal{C}(a)$ which is a contradiction. Hence $\mu(a) = \mu(b)$. ■

Theorem 3.11. Let μ be a fuzzy filter in a residuated lattice L. Then $\pi(\mu)$ is a fuzzy lattice filter in $A/\cong$ and $\pi^{-1}(\pi(\mu)) = \mu$.

Proof: Let $[a], [b] \in A/\cong$. Then $\pi(a) = [a]$ and $\pi(b) = [b]$. Since π is a homomorphism, we have $[a] \wedge [b] = [a \wedge b] = \pi(a \wedge b)$. We get that $a \wedge b = \pi^{-1}(x \wedge y)$. We have

$$\begin{aligned}
\pi(\mu)([a] \wedge [b]) &= \sup\{\mu(z) : z \in \pi^{-1}[a \wedge b]\} \\
&\geq \sup\{\mu(x \wedge y) : x \in \pi^{-1}([a]), y \in \pi^{-1}([b])\} \\
&= \sup\{\min\{\mu(x), \mu(y)\} : x \in \pi^{-1}([a]), y \in \pi^{-1}([b])\} \\
&= \min\{\sup\{\mu(x) : x \in \pi^{-1}([a])\}, \sup\{\mu(b) : y \in \pi^{-1}([b])\}\} \\
&= \min\{\pi(\mu)(a), \pi(\mu)(b)\}.
\end{aligned}$$

Let $[a] \leq [b]$. Then $\pi(a) \leq \pi(b)$. We shall show that $\pi(\mu)([a]) \leq \pi(\mu)([b])$. Suppose that $\pi(\mu)[a] > \pi(\mu)[b]$. Then there exists $x_0 \in \pi^{-1}([a])$ such that $\pi(x_0) = a$ and $\mu(x_0) > \sup\{\mu(y) : y \in \pi^{-1}(b)\}$. We have $\mu(y) \leq \mu(x_0)$ for all $y \in \pi^{-1}(b)$. Let $y \in \pi^{-1}(b)$ be arbitrary. Since π is a lattice homomorphism, then $[b] = [a] \vee [b] = \pi(x_0) \vee \pi(y) = \pi(x_0 \vee y)$. Hence $x_0 \vee y \in \pi^{-1}(b)$. Therefore $\mu(x_0 \vee y) < \mu(x_0)$. By Definition 2.17, $\mu(x_0 \vee y) \geq \max\{\mu(x_0), \mu(y)\} = \mu(x_0)$ which is a contradiction. Hence $\pi(\mu)$ is a fuzzy lattice filter in $A/\cong$. By Lemma 3. 10, we have $\pi^{-1}(\pi(\mu))(a) = \pi(\mu)(\pi(a)) = \pi(\mu)[a] = \sup\{\mu(x) : x \in \pi^{-1}([a])\} = \sup\{\mu(x) : \pi(x) = [a]\} = \sup\{\mu(x) : [x] = [a]\} = \mu(a)$.■

Theorem 3.12. Let ν be a fuzzy lattice filter in a lattice $A/\cong$. Then $\pi^{-1}(\nu)$ is a fuzzy filter in A and $\pi(\pi^{-1}(\nu)) = \nu$.

Proof: Let $x, y \in A$. By Lemma 3.5 part (iii), we have $\pi^{-1}(\nu)(x * y) = \nu(\pi(x * y)) = \nu([x*y]) = \nu([x \wedge y]) = \nu([x] \wedge [y]) \geq \min\{\nu([x]), \nu([y])\} = \min\{\pi^{-1}(\nu)(x), \pi^{-1}(\nu)(y)\}$. Suppose that $x \leq y$. By Lemma 3.5 part (ii), we have $[x] \leq [y]$. Since ν is a fuzzy lattice filter in $A/\cong$, we have $\nu([x]) \leq \nu([y])$, that is $\pi^{-1}(\nu)(x) \leq \pi^{-1}(\nu)(y)$. By Lemma 3.10, we obtain that $\pi(\pi^{-1}(\nu))[x] = \sup\{\pi^{-1}(\nu)(y) : y \in \pi^{-1}([x])\} = \sup\{\pi^{-1}(\nu)(y) : \pi(y) = [x]\} = \sup\{\pi^{-1}(\nu)(y) : [y] = [x]\} = \nu([x])$.■

Proposition 3.13. Let μ and ν be fuzzy filters in a residuated lattice A. Then $\nu \leq \mu$ if and only if $\pi(\nu) \leq \pi(\mu)$.

Proof: Suppose that $\nu \leq \mu$. Then $\pi(\nu)([x]) = \sup\{\nu(y) : y \in \pi^{-1}([x])\} \leq \sup\{\mu(y) : y \in \pi^{-1}([x])\} = \pi(\mu)([x])$. Conversely, let $\pi(\nu) \leq \pi(\mu)$. Then $\nu(a) = \pi^{-1}(\pi(\nu))(a) = \pi(\nu)(\pi(a)) \leq \pi(\mu)(\pi(a)) = \pi^{-1}(\pi(\mu))(a) = \mu(a)$.■

Theorem 3.14. There is a lattice isomorphism between the lattices $\mathcal{F}(A)$ and $\mathcal{FL}(A/\cong)$.

Proof: Define $\varphi : \mathcal{F}(A) \to \mathcal{FL}(A/\cong)$ by $\varphi(\mu) = \pi(\mu)$ and $\psi : \mathcal{F}(L/\equiv) \to \mathcal{F}(L)$ by $\psi(\nu) = \pi^{-1}(\nu)$. By Theorems 3.11 and 3.12 φ and ψ are well defined and bijection. By the above Proposition φ is a lattice homomorphism. Hence ϕ is an isomorphism of lattices.■

Theorem 3.15. Let μ be a fuzzy prime filter in a residuated lattice A. Then $\pi(\mu)$ is a fuzzy prime filter in $A/\cong$.

Proof: Since μ is a fuzzy prime filter in A, then μ is proper. So $\mu(0) \neq \mu(1)$. By Lemma 3.10, $\pi(\mu)(0) = \sup\{\mu(x) : x \in \pi^{-1}([0])\} = \sup\{\mu(x) : [x] = [0]\} = \mu(0) = 0$ and $\pi(\mu)(1) = \sup\{\mu(x) : x \in \pi^{-1}([1])\} = \sup\{\mu(x) : [x] = [1]\} = \mu(1) = 1$. Hence $\pi(\mu)$ is proper. We have $\pi(\mu)([x \vee y]) = \sup\{\mu(z) : z \in \pi^{-1}([x \vee y])\} = \sup\{\mu(x) : [z] = [x \vee y]\} = \mu(x \vee y) = \max\{\mu(x), \mu(y)\}$ Also, we have $\pi(\mu)[x] = \sup\{\mu(a) : a \in \pi^{-1}([x])\}. = \sup\{\mu(a) : [a] = [x]\} = \mu(x)$. Similarly, we can show that $\pi(\mu)[y] = \mu(y)$. We obtain that $\pi(\mu)(x \vee y) = \mu(x \vee y) = \max\{\mu(x), \mu(y)\} = \max\{\pi(\mu)(x), \pi(\mu)(y)\}$ and then $\pi(\mu)$ is a fuzzy prime filter in $A/\cong$.■

Theorem 3.16. Let ν be a fuzzy prime filter in a lattice $A/\cong$. Then $\pi^{-1}(\nu)$ is a fuzzy prime filter in A.

Proof: By assumption ν is proper. Hence $\nu([0]) \neq \nu([1])$. We have $\pi^{-1}(\nu)(0) = \nu(\pi(0)) = \nu([0])$ and $\pi^{-1}(\nu)(1) = \nu(\pi(1)) = \nu([1])$. Hence $\pi^{-1}(\nu)(0) \neq \pi^{-1}(\nu)(1)$. That is $\pi^{-1}(\nu)$ is proper. Also, we have
$\pi^{-1}(\nu)(x \vee y) = \nu(\pi(x \vee y)) = \nu([x \vee y]) = \nu([x] \vee [y]) = \max\{\nu([x]), \nu([y])\} = \max\{\pi^{-1}(\nu)(x), \pi^{-1}(\nu)(y)\}$. ■

Theorem 3.17. There exists a homomorphism between topological Space $Fspec(A)$ and $Fspec(A/\cong)$.

Proof: Consider φ in Theorem 3.14. The restriction φ to $Fspec(A)$ is denoted by $\bar{\varphi}$. By Theorems 3.15 and 3.16, $\bar{\varphi} : Fspec(A) \to Fspec(A/\equiv)$ is a bijective. We will show that $\bar{\varphi}$ is continuous and closed. Let $\mathcal{C}([a])$ be an arbitrary closed base set. Then

$$\begin{aligned}\bar{\varphi}^{-1}(\mathcal{C}([a]) &= \{\mu \in Fspec(A) : \bar{\varphi}(\mu) \in \mathcal{C}([a])\}\\ &= \{\mu \in Fspec(A) : \pi(\mu) \in \mathcal{C}([a])\}\\ &= \{\mu \in Fspec(A) : \pi(\mu)[a] = 1\}\\ &= \{\mu \in Fspec(A) : \mu(a) = 1\} = \mathcal{C}(a).\end{aligned}$$

Hence φ is continuous. Also, we have

$$\begin{aligned}\bar{\varphi}(\mathcal{C}(a)) &= \{\varphi(\mu) : \mu \in Fspec(A), \mu \in \mathcal{C}(a)\}\\ &= \{\pi(\mu) : \mu \in Fspec(A), \mu \in \mathcal{C}(a)\}\\ &= \{\pi(\mu) : \mu \in Fspec(A), \mu(a) = 1\}\\ &= \{\nu \in Fspec(A/\cong) : \nu([a]) = 1\} = \mathcal{C}([a]).\end{aligned}$$

Hence φ is closed.■

Let A be a residuated lattice. For any $a, b \in A$ define $a \equiv b$ iff for any $P \in Spec(A)$, ($a \in P$ iff $b \in P$). Then $\equiv$ is a congruence relation on A respect to $*$, $\wedge$ and $\vee$. Let us denot by $\bar{a}$ the equivalence class of $a \in A$ and let $A/\equiv$ be the quotient set. We denote $\lambda : A \to A/\equiv$ the canonical surjective defined by $\lambda(a) = \bar{a}$. Then $(A/\equiv, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $(A/\equiv, \lambda)$ is a reticulation of A (See [8]).

Theorem 3.18. Let A be a residuated lattice. Then the congruence relation $\cong$ is equal to the congruence relation $\equiv$ on A.

Proof: Let $a, b \in A$ such that $a \cong b$. We have ($\mu(a) = 1$ iff $\mu(b) = 1$) for any $\mu \in Fspec(A)$. Suppose that $P \in Spec(A)$. By Theorem 2.18, χ_P is a fuzzy prime filter. Hence $\chi_P(a) = 1$ iff $\chi_P(b) = 1$. We get that $a \in P$ iff $b \in P$. Hence $a \equiv b$ and then $\cong \subseteq \equiv$.
Conversely, let $a \equiv b$ and $\mu \in Fspec(A)$ such that $\mu(a) = 1$. We get that $a \in \mu_1$ and μ_1 is a proper filter of A. Hence $\mu_1 \in Spec(A)$. Since $a \equiv b$, then we have $b \in \mu_1$. We obtain that $\mu(b) = 1$. Similarly, we can prove that if $\mu(b) = 1$, then $\mu(b) = 1$. So $a \cong b$. Therefor $\equiv \subseteq \cong$.■

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