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Instability to nonlinear vector differential equations of fifth order with constant delay

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ABSTRACT: We consider a certain vector differential equation of the fifth order with a constant delay. We give new certain sufficient conditions which guarantee the instability of the zero solution of that equation. An example is given to illustrate the theoretical analysis made in the paper.

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1. Introduction

In 2003, Sadek [5] considered the nonlinear vector differential equation of the fifth order:

$$X^{(5)} + \Psi(\ddot{X})\dddot{X} + \Phi(\ddot{X}) + \Theta(\dot{X}) + F(X) = 0. \tag{1.1}$$

The author gave certain sufficient conditions, which guarantee the instability of the zero solution of Eq. (1.1).

In this paper, instead of Eq. (1.1), we consider its delay form as follows:

$$X^{(5)} + \Psi(\ddot{X})\dddot{X} + \Phi(X, \dot{X}, \ddot{X})\ddot{X} + H(\dot{X}(t-\tau)) + F(X(t-\tau)) = 0, \tag{1.2}$$

where $X \in \Re^n$, $\tau > 0$ is the constant deviating argument, Ψ and Φ are continuous $n \times n$ -symmetric matrix functions for the arguments displayed explicitly, $H : \Re^n \to \Re^n$ and $F : \Re^n \to \Re^n$ with $H(0) = F(0) = 0$, and H and F are continuous functions for the arguments displayed explicitly. It is assumed the existence and the uniqueness of the solutions of Eq. (1.2).

Eq. (1.2) is the vector version for systems of real nonlinear differential equations of the fifth order:

$$\begin{aligned} x_i^{(5)} \quad &+ \quad \sum_{k=1}^{n} \psi_{ik}(x_1'', , ..., x_k'')x_k''' + \sum_{k=1}^{n} \phi_{ik}(x_1, , ..., x_k, ..., x_1'', , ..., x_k'')x_k'' \\ &+ \quad h_i(x_1'(t-\tau), ..., x_n'(t-\tau)) + f_i(x_1(t-\tau), ..., x_n(t-\tau)) = 0, \end{aligned}$$

for $i = 1, 2, ..., n$.

Instead of Eq. (1.2), we consider its equivalent differential system

$$\begin{aligned}
\dot{X} &= Y, \dot{Y} = Z, \dot{Z} = W, \dot{W} = U, \\
\dot{U} &= -\Psi(Z)W - \Phi(X,Y,Z)Z - H(Y) - F(X) \\
&\quad + \int\limits_{t-\tau}^{t} J_H(Y(s))Z(s)ds + \int\limits_{t-\tau}^{t} J_F(X(s))Y(s)ds,
\end{aligned} \tag{1.3}$$

which was obtained by setting $\dot{X} = Y$, $\ddot{X} = Z$, $\dddot{X} = W$ and $X^{(4)} = U$ from Eq. (1.2).

$J_F(X)$ and $J_H(Y)$ denote the Jacobian matrices corresponding to the functions $F(X)$ and $H(Y)$, respectively. It is clear that

$$J_F(X) = \left(\frac{\partial f_i}{\partial x_j}\right)$$

and

$$J_H(Y) = \left(\frac{\partial h_i}{\partial y_j}\right), (i,\ j = 1, 2, ..., n),$$

where $(x_1, ..., x_n)$, $(y_1, ..., y_n)$, $(f_1, ..., f_n)$ and $(h_1, ..., h_n)$ are components of X, Y, F and H, respectively. Throughout what follows, it is assumed that $J_F(X)$ and $J_H(Y)$ exist and are symmetric and continuous.

It should be noted that since 1978 till now the instability of the solutions of certain scalar differential equations of the fifth order without and with delay and vector differential equations of the fifth order without delay was discussed in the literature. For a comprehensive treatment of the subject, we refer the readers to the papers of Ezeilo [2], Sadek [5], Sun and Hou [6], Tunç ([7]-[14]), Tunç and Erdogan [15], Tunç and Karta [16], Tunç and Şevli [17] and the references cited in these sources. However, a review to date of the literature indicates that the instability of solutions of vector differential equations of the fifth order with delay has not been investigated. This paper is the first known publication regarding the instability of solutions for the nonlinear vector differential equations of the fifth order with a deviating argument. The motivation of this paper comes from the above papers done on scalar differential equations of the fifth order without and with delay and the vector differential equations of the fifth order without delay. Our aim is to achieve the results established in Sadek [[5], Theorem 3] to Eq. (1.2). By this work, we improve the results of Sadek [[5], Theorem 3] to a vector differential equation of the fifth order with delay. Based on Krasovskii's criterions [3], we prove our main result, and an example is also provided to illustrate the feasibility of the proposed result. The result to be obtained is new and different from that in the papers mentioned above.

Note that the instability criteria of Krasovskii [3] can be summarized as the following: According to these criteria, it is necessary to show here that there exists a Lyapunov- Krasovskii functional $V(.) \equiv V(X,Y,Z,W,U)$ which has Krasovskii properties, say (K_1), (K_2) and (K_3) :

(K_1) In every neighborhood of $(0,0,0,0,0)$, there exists a point $(\xi_1,...,\xi_5)$ such that $V(\xi_1,...,\xi_5)>0$,

(K_2) the time derivative $\frac{d}{dt}V(.)$ along solution paths of (1.3) is positive semi-definite,

(K_3) the only solution $(X,Y,Z,W,U)=(X(t),Y(t),Z(t),W(t),U(t))$ of (1.3) which satisfies $\frac{d}{dt}V(.)=0$, $(t\geq 0)$, is the trivial solution $(0,0,0,0,0)$.

The symbol $\langle X,Y\rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum\limits_{i=1}^{n} x_i y_i$, that is, $\langle X,Y\rangle=\sum\limits_{i=1}^{n} x_i y_i$; thus $\langle X,X\rangle=\|X\|^2$, and $\lambda_i(\Omega)$, $(i=1,2,...,n)$, are the eigenvalues of the real symmetric $n\times n$ - matrix Ω. The matrix Ω is said to be negative-definite, when $\langle \Omega X,X\rangle\leq 0$ for all nonzero X in $\Re^n$.

2. Main results

Before introduction of the main result, we need the following results.

Lemma 2.1. *(Bellman [1]). Let A be a real symmetric $n\times n$ -matrix and*

$$a'\geq \lambda_i(A)\geq a>0, (i=1,2,...,n),$$

where a' and a are constants.

Then

$$a'\langle X,X\rangle\geq\langle AX,X\rangle\geq a\langle X,X\rangle$$

and

$$a'^2\langle X,X\rangle\geq\langle AX,AX\rangle\geq a^2\langle X,X\rangle.$$

The following theorem, due to the Russian mathematician N. G. Četaev's (LaSalle and Lefschetz [4]).

Theorem 2.1. *(Instability Theorem of Četaev's). Let Ω be a neighborhood of the origin. Let there be given a function $V(x)$ and region Ω_1 in Ω with the following properties:*

(i) $V(x)$ has continuous first partial derivatives in Ω_1.

(ii) $V(x)$ and $\dot{V}(x)$ are positive in Ω_1.

(iii) At the boundary points of Ω_1 inside Ω, $V(x)=0$.

(iv) The origin is a boundary point of Ω_1.

Under these conditions the origin is unstable.

Let $r\geq 0$ be given, and let $C=C([-r,0],\ \Re^n)$ with

$$\|\phi\|=\max_{-r\leq s\leq 0}|\phi(s)|,\quad \phi\in C.$$

For $H>0$ define $C_H\subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r,\ A) \to \Re^n$ is continuous, $0 < A \leq \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t+s), -r \leq s \leq 0, t \geq 0.$$

Let G be an open subset of C and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), F(0) = 0, x_t = x(t+\theta), -r \leq \theta \leq 0, t \geq 0,$$

where $F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on F that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition 2.2. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.

The main result of this paper is the following theorem.

Theorem 2.2. *In addition to the basic assumptions imposed on* Ψ, Φ, H *and* F *that appear in Eq. (1.2), we assume that there exist positive constants* a_3, a_4 *and* a_5 *such that the following conditions hold:*

$$\Psi(Z),\ \Phi(X,Y,Z), J_H(Y) \text{ and } J_F(X) \text{ are symmetric},$$

$$F(0) = 0, F(X) \neq 0, (X \neq 0), \lambda_i(J_F(X)) \leq -a_5$$
$$H(0) = 0, H(Y) \neq 0, (Y \neq 0), |\lambda_i(J_H(Y))| \leq a_4$$

and

$$\lambda_i(\Phi(X,Y,Z)) \geq a_3 \quad \text{for all } X \in \Re^n.$$

If

$$\tau < \min\{\frac{2}{\sqrt{n}}, \frac{2a_3}{2\sqrt{n}a_4 + \sqrt{n}a_5}\},$$

then the zero solution of Eq. (1.2) is unstable.

Remark 2.3. It should be noted that there is no sign restriction on eigenvalues of Ψ, and it is clear that our assumptions have a very simple form and the applicability of them can be easily verified.

Proof. We define a Lyapunov-Krasovskii functional

$$
\begin{aligned}
V(.) = V(X(t), Y(t), Z(t), W(t), U(t)) :& \\
V(.) =& -\langle Y, F(X)\rangle - \langle Z, U\rangle + \frac{1}{2}\langle W, W\rangle \\
&- \int_0^1 \langle H(\sigma Y), Y\rangle\, d\sigma - \int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle\, d\sigma \\
&- \lambda \int_{-\tau}^{0} \int_{t+s}^{t} \|Y(\theta)\|^2\, d\theta ds - \mu \int_{-\tau}^{0} \int_{t+s}^{t} \|Z(\theta)\|^2\, d\theta ds,
\end{aligned}
$$

where λ and μ are certain positive constants; the constants λ and μ will be determined later in the proof.

It is clear that $V(0, 0, 0, 0, 0) = 0$ and

$$
V(0, 0, 0, \varepsilon, 0) = \frac{1}{2}\langle \varepsilon, \varepsilon\rangle = \frac{1}{2}\|\varepsilon\|^2 > 0
$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in \Re^n$, which verifies the property (P_1) of Krasovskii [3].

Using a basic calculation,the time derivative of $V(.)$ along solutions of (1.3) results in

$$
\begin{aligned}
\frac{d}{dt}V(.) = & - \langle Y, J_F(X)Y\rangle + \langle \Psi(Z)W, Z\rangle + \langle Z, \Phi(X, Y, Z)Z\rangle \\
& + \langle H(Y), Z\rangle + < \int_{t-\tau}^{t} J_F(X(s))Y(s)ds, Z > \\
& + < \int_{t-\tau}^{t} J_H(Y(s))Z(s)ds, Z > \\
& - \frac{d}{dt}\int_0^1 \langle H(\sigma Y), Y\rangle\, d\sigma - \frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle\, d\sigma \\
& - \lambda\frac{d}{dt}\int_{-\tau}^{0} \int_{t+s}^{t} \|Y(\theta)\|^2\, d\theta ds - \mu\frac{d}{dt}\int_{-\tau}^{0} \int_{t+s}^{t} \|Z(\theta)\|^2\, d\theta ds.
\end{aligned}
$$

It can be easily seen that

$$\frac{d}{dt}\int_0^1 \langle H(\sigma Y), Y\rangle \, d\sigma = \langle H(Y), Z\rangle,$$

$$\frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle \, d\sigma = \langle \Psi(Z)W, Z\rangle,$$

$$\frac{d}{dt}\int_{-\tau}^{0}\int_{t+s}^{t} \|Y(\theta)\|^2 \, d\theta ds = \|Y\|^2\,\tau - \int_{t-\tau}^{t} \|Y(\theta)\|^2 \, d\theta,$$

$$\frac{d}{dt}\int_{-\tau}^{0}\int_{t+s}^{t} \|Z(\theta)\|^2 \, d\theta ds = \|Z\|^2\,\tau - \int_{t-\tau}^{t} \|Z(\theta)\|^2 \, d\theta,$$

$$\begin{aligned}
< \int_{t-\tau}^{t} J_F(X(s))Y(s)ds, Z > &\geq -\|Z\| \left\| \int_{t-\tau}^{t} J_F(X(s))Y(s)ds \right\| \\
&\geq -\sqrt{n}a_5 \|Z\| \left\| \int_{t-\tau}^{t} Y(s)ds \right\| \\
&\geq -\sqrt{n}a_5 \|Z\| \int_{t-\tau}^{t} \|Y(s)\| \, ds \\
&\geq -\frac{1}{2}\sqrt{n}a_5\tau \|Z\|^2 - \frac{1}{2}\sqrt{n}a_5 \int_{t-\tau}^{t} \|Y(s)\|^2 \, ds
\end{aligned}$$

and

$$\begin{aligned}
< \int_{t-\tau}^{t} J_H(Y(s))Z(s)ds, Z > &\geq -\|Z\| \left\| \int_{t-\tau}^{t} J_H(Y(s))Z(s)ds \right\| \\
&\geq -\sqrt{n}a_4 \|Z\| \left\| \int_{t-\tau}^{t} Z(s)ds \right\| \\
&\geq -\sqrt{n}a_4 \|Z\| \int_{t-\tau}^{t} \|Z(s)\| \, ds \\
&\geq -\frac{1}{2}\sqrt{n}a_4\tau \|Z\|^2 - \frac{1}{2}\sqrt{n}a_4 \int_{t-\tau}^{t} \|Z(s)\|^2 \, ds
\end{aligned}$$

so that

$$\begin{aligned}
\frac{d}{dt}V(.) \quad \geq \quad & -\langle Y, J_F(X)Y\rangle + \langle Z, \Phi(X,Y,Z)Z\rangle \\
& -\frac{1}{2}\sqrt{n}a_5\tau\langle Z, Z\rangle \;\; -\frac{1}{2}\sqrt{n}a_5 \int\limits_{t-\tau}^{t} \|Y(s)\|^2\, ds \\
& -\frac{1}{2}\sqrt{n}a_4\tau\langle Z, Z\rangle \;\; -\frac{1}{2}\sqrt{n}a_4 \int\limits_{t-\tau}^{t} \|Z(s)\|^2\, ds \\
& -\lambda\tau\langle Y, Y\rangle + \lambda \int\limits_{t-\tau}^{t} \|Y(\theta)\|^2\, d\theta \\
& -\mu\tau\langle Z, Z\rangle + \mu \int\limits_{t-\tau}^{t} \|Z(\theta)\|^2\, d\theta \\
\geq \quad & (a_5 - \lambda\tau)\|Y\|^2 \\
& +\{a_3 - (\mu + \frac{1}{2}\sqrt{n}a_4 + \frac{1}{2}\sqrt{n}a_5)\tau\}\|Z\|^2 \\
& +(\lambda - \frac{1}{2}\sqrt{n}a_5) \int\limits_{t-\tau}^{t} \|Y(s)\|^2\, ds \\
& +(\mu - \frac{1}{2}\sqrt{n}a_4) \int\limits_{t-\tau}^{t} \|Z(s)\|^2\, ds.
\end{aligned}$$

Let

$$\lambda = \frac{1}{2}\sqrt{n}a_5$$

and

$$\mu = \frac{1}{2}\sqrt{n}a_4$$

so that

$$\frac{d}{dt}V(.) \geq \{(a_5 - \frac{1}{2}\sqrt{n}a_5)\tau\}\|Y\|^2 + \{(a_3 - (\sqrt{n}a_4 + \frac{1}{2}\sqrt{n}a_5)\tau\}\|Z\|^2.$$

If $\tau < \min\{\frac{2}{\sqrt{n}}, \frac{2a_3}{2\sqrt{n}a_4 + \sqrt{n}a_5}\}$, then we have for some positive constants k_1 and k_2 that

$$\frac{d}{dt}V(.) \geq k_1\|Y\|^2 + k_2\|Z\|^2 \geq 0,$$

which verifies the property (P_2) of Krasovskii [3].

On the other hand, it follows that

$$\frac{d}{dt}V(.) = 0 \Leftrightarrow Y = \dot{X} = 0, Z = \dot{Y} = 0, W = \dot{Z} = 0, U = \dot{W} = 0 \quad \text{for all } t \geq 0.$$

Hence

$$X = \xi, Y = Z = W = U = 0,$$

where ξ is a constant vector.

Substituting foregoing estimates in the system (1.3), we get that $F(\xi) = 0$, which necessarily implies that $\xi = 0$ since $F(0) = 0$. Thus, we have

$$X = Y = Z = W = U = 0 \quad \text{for all } t \geq 0.$$

Hence, the property (P_3) of Krasovskii [3] holds

The proof of Theorem 2.2 is complete. □

Example 2.4. In a special case of Eq. (1.2), for $n = 2$, we choose

$$\Psi(Z) = \begin{bmatrix} z_1 & 1 \\ 1 & z_2 \end{bmatrix},$$

$$\Phi(X, Y, Z) = \begin{bmatrix} 9 + \frac{1}{1+x_1^2+y_1^2+z_1^2} & 0 \\ 0 & 9 + \frac{1}{1+x_2^2+y_2^2+z_2^2} \end{bmatrix},$$

$$H(Y(t-\tau)) = \begin{bmatrix} 4y_1(t-\tau) \\ 4y_2(t-\tau) \end{bmatrix}$$

and

$$F(X(t-\tau)) = \begin{bmatrix} -3x_1(t-\tau) \\ -3x_2(t-\tau) \end{bmatrix}.$$

Then, the matrix $\Psi(Z)$ is symmetric, and, by an easy calculation, we obtain

$$\lambda_1(\Phi(X, Y, Z)) = 9 + \frac{1}{1 + x_1^2 + y_1^2 + z_1^2},$$

$$\lambda_2(\Phi(X, Y, Z)) = 9 + \frac{1}{1 + x_2^2 + y_2^2 + z_2^2},$$

$$J_H(Y) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

and

$$J_F(X) = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

so that

$$\lambda_i(\Phi(X, Y, Z)) \geq 9 = a_3 > 0,$$

$$|\lambda_i(J_H(Y))| = 4 = a_4$$

and

$$\lambda_i(J_F(X)) \le -3 = -a_5, \quad (i = 1,\ 2).$$

Thus, if

$$\tau < \min\{\frac{2}{\sqrt{2}}, \frac{8}{8\sqrt{2}+3\sqrt{2}}\},$$

then all the assumptions of Theorem 2.2 hold.

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References

[1] Bellman, R., Introduction to matrix analysis. Reprint of the second (1970) edition. With a foreword by Gene Golub. Classics in Applied Mathematics, 19. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

[2] Ezeilo, J. O.C., Instability theorems for certain fifth-order differential equations, *Math. Proc. Cambridge Philos. Soc.* 84 (1978), no. 2, 343–350.

[3] Krasovskii, N. N., On conditions of inversion of A. M. Lyapunov's theorems on instability for stationary systems of differential equations (Russian), *Dokl. Akad. Nauk. SSSR (N.S.)*, 101, (1955), 17–20.

[4] LaSalle, J.& Lefschetz, S., Stability by Liapunov's direct method with applications. Mathematics in Science and Engineering, Vol. 4, Academic Press, New York-London, 1961.

[5] Sadek, A. I., Instability results for certain systems of fourth and fifth order differential equations, *Appl. Math. Comput.* 145 (2003), no. 2-3, 541–549.

[6] Sun, Wu Jun; Hou, Xia, New results about instability of some fourth and fifth order nonlinear systems (Chinese), *J. Xinjiang Univ. Natur. Sci.* 16 (1999), no. 4, 14–17.

[7] Tunç, C., Further results on the instability of solutions of certain nonlinear vector differential equations of fifth order, *Appl. Math. Inf. Sci.* 2 (2008), no. 1, 51–60.

[8] Tunç, C., On the instability of solutions of some fifth order nonlinear delay differential equations, A*ppl. Math. Inf. Sci. (AMIS).* 5 (2011), no.1, 112–121.

[9] Tunç, C., An instability theorem for a certain fifth-order delay differential equation, *Filomat* 25:3 (2011), 145–151.

[10] Tunç, C., Recent advances on instability of solutions of fourth and fifth order delay differential equations with some open problems. World Scientific Review, Vol. 9, World Scientific Series on Nonlinear Science Series B (Book Series), (2011), 105–116.

[11] Tunç, C., On the instability of solutions of nonlinear delay differential equations of fourth and fifth order, *Sains Malaysiana* 40 (12), (2011), 1455–1459.

[12] Tunç, C., Instability for nonlinear differential equations of fifth order subject to delay, *Nonlinear Dyn. Syst. Theory* 12 (2) (2012), 207–214.

[13] Tunç, C., Instability of a nonlinear differential equation of fifth order with variable delay, *Int. J. Comput. Math. Sci.* 6 (2012), 73–75.

[14] Tunç, C., Instability of solutions for nonlinear functional differential equations of fifth order with n-deviating arguments, *Bul. Acad. Ştiinte Repub. Mold. Mat.* 68 (2012), no. 1, 3–14.

[15] Tunç, C.; Erdogan, F., On the instability of solutions of certain non-autonomous vector differential equations of fifth order, *SUT J. Math.* 43 (2007), no. 1, 35–48.

[16] Tunç, C.; Karta, M., A new instability result to nonlinear vector differential equations of fifth order, *Discrete Dyn. Nat. Soc.* 2008, Art. ID 971534, 6 pp.

[17] Tunç, C.; Şevli, H., On the instability of solutions of certain fifth order nonlinear differential equations, *Mem. Differential Equations Math. Phys.* 35 (2005), 147–156.

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