Journal of
Mathematics
and Applications

JMA No 36, pp 113-120 (2009)

Instability to differential equations of fourth order with a variable deviating argument

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Abstract: The main purpose of this paper is to give two instability theorems to fourth order nonlinear differential equations with a variable deviating argument.

AMS Subject Classification: *34K20*
Keywords and Phrases: *Instability; Krasovskii criteria; differential equation; fourth order; deviating argument*

1. Introduction

In 2000, Ezeilo [5] proved two instability theorems for the fourth order nonlinear differential equations without delay

$$x^{(4)} + a_1 x''' + g(x, x', x'', x''')x'' + h(x)x' + f(x, x', x'', x''') = 0 \tag{1.1}$$

and

$$x^{(4)} + p(x''', x'') + q(x, x', x'', x''')x'' + a_3 x' + a_4 x = 0. \tag{1.2}$$

In this paper, instead of Eq. (1.1) and Eq. (1.2), we consider the fourth order nonlinear differential equations with a variable deviating argument, $\tau(t)$:

$$\begin{aligned} x^{(4)}(t) &+ a_1 x'''(t) + g(x(t-\tau(t)), ..., x'''(t-\tau(t)))x'' \\ &+ h(x(t))x'(t) + f(x(t-\tau(t)), ..., x'''(t-\tau(t)))x(t) = 0 \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} x^{(4)}(t) &+ p(x'''(t), x''(t)) + q(x(t-\tau(t)), ..., x'''(t-\tau(t)))x'' \\ &+ a_3 x'(t) + a_4 x(t) = 0. \end{aligned} \tag{1.4}$$

We write Eq. (1.3) and Eq. (1.4) in system form as

$$
\begin{aligned}
x' &= y, \\
y' &= z, \\
z' &= u, \\
u' &= -a_1 u - g(x(t-\tau(t)), ..., u(t-\tau(t)))z - h(x)y \\
&\quad -f(x(t-\tau(t)), ..., u(t-\tau(t)))x
\end{aligned}
\tag{1.5}
$$

and

$$
\begin{aligned}
x' &= y, \\
y' &= z, \\
z' &= u, \\
u' &= -p(u,z) - q(x(t-\tau(t)), ..., u(t-\tau(t)))z \\
&\quad -a_3 y - a_4 x,
\end{aligned}
\tag{1.6}
$$

respectively, where $\tau(t)$ is fixed delay, $t-\tau(t)$ is strictly increasing, $\lim\limits_{t\to\infty}(t-\tau(t)) = \infty$, $t \in \Re_+ = [0,\infty)$; $a_1,\ a_3$ and a_4 are constants; $g,\ h,\ f,\ p$ and q are continuous functions in their respective arguments on $\Re^4$, $\Re$, $\Re^4$, $\Re^2$ and $\Re^4$, respectively, with $p(0,z) = 0$ and satisfy a Lipschitz condition in their respective arguments; the derivative $\frac{\partial p}{\partial z}(u,z)$ exists and is also continuous. Hence, the existence and uniqueness of the solutions of Eq. (1.3) and Eq. (1.4) are guaranteed (see [[2], pp.14]). We assume in what follows that $x(t),\ y(t),\ z(t)$ and $u(t)$ are abbreviated as x, y, z and u , respectively.

So far, the instability of solutions to certain fourth order nonlinear scalar and vector differential equations without delay has been investigated by many authors (see Dong and Zhang [1], Ezeilo ([3]-[5]), Li and Duan [8], Li and Yu [9], Lu and Liao [10], Sadek [11], Skrapek [12], Sun and Hou [13], Tiryaki [14], Tunç ([15]-[18]), C. Tunç and E. Tunç [20] and the references cited thereof). However, by now, the instability of solutions to fourth order nonlinear differential equations with deviating arguments has only been studied by Tunç [19]. This paper is the second attempt on the topic in the literature. It is worth mentioning that throughout all of the papers, based on Krasovskii's properties (see Krasovskii [6]), the Lyapunov's second (or direct) method has been used as a basic tool to prove the results established therein. The motivation for this paper comes from the above mentioned papers. Our aim is to carry out the results established in Ezeilo [5] to nonlinear differential equations of fourth order, Eq. (1.3) and Eq. (1.4), with a deviating argument for the instability of zero solution of these equations.

Note that the instability criteria of Krasovskii [6] can be summarized as the following: According to these criteria, it is necessary to show here that there exists a Lyapunov function $V(.) \equiv V(x,y,z,u)$ which has Krasovskii properties, say (K_1), (K_2) and (K_3) :

(K_1) In every neighborhood of $(0,0,0,0)$, there exists a point (ξ,η,ζ,μ) such that $V(\xi,\eta,\zeta,\mu) > 0$;

(K_2) the time derivative $\dot{V} = \frac{d}{dt}V(x, y, z, u)$ along solution paths of the system (1.5) is positive semi-definite;

(K_3) the only solution $(x, y, z, u) = (x(t), y(t), z(t), u(t))$ of the system (1.5) which satisfies $\dot{V} = 0$, $(t \geq 0)$, is the trivial solution $(0, 0, 0, 0)$.

Let $r \geq 0$ be given, and let $C = C([-r, 0], \ \Re^n)$ with

$$\|\phi\| = \max_{-r \leq s \leq 0} |\phi(s)|, \quad \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r, \ A) \to \Re^n$ is continuous, $0 < A \leq \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t + s), -r \leq s \leq 0, t \geq 0.$$

Let G be an open subset of C and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), x_t = x(t + \theta), -r \leq \theta \leq 0, t \geq 0,$$

where $F(0) = 0$ and $F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on F that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition 1.1. Let $F(0) = 0$. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.

Theorem 1.1. *(Instability Theorem of Cetaev's). Let Ω be a neighborhood of the origin. Let there be given a function $V(x)$ and region Ω_1 in Ω with the following properties:*

(i) *$V(x)$ has continuous first partial derivatives in Ω_1.*

(ii) *$V(x)$ and $\dot{V}(x)$ are positive in Ω_1.*

(iii) *At the boundary points of Ω_1 inside Ω, $V(x) = 0$.*

(iv) *The origin is a boundary point of Ω_1.*

Under these conditions the origin is unstable (see LaSalle and Lefschetz [7]).

2. Main results

The first main result is the following theorem.

Theorem 2.1. *Suppose that*

$$f(x(t-\tau(t)),...,u(t-\tau(t))) - \frac{1}{4}g^2(x(t-\tau(t)),...,u(t-\tau(t))) > 0$$

for arbitrary $x(t-\tau(t)),...,\ u(t-\tau(t))$*. Then the zero solution of Eq. (1.3) is unstable.*

Proof. Consider the Lyapunov function $V = V(x, y, z, u)$ defined by

$$V = yz + \frac{1}{2}a_1y^2 - xu - a_1xz - \int_0^x h(s)sds, \quad \text{(where } a_1 \text{ is a constant)},$$

so that

$$V(0, \varepsilon^2, \varepsilon, 0) = \varepsilon^3 + \frac{1}{2}a_1\varepsilon^4 > 0$$

for sufficiently small ε. In fact, if ε is an arbitrary positive constant, then

$$V(0, \varepsilon^2, \varepsilon, 0) > 0$$

for sufficiently small ε. Thus V satisfies the property (K_1), (see [6]).

By an elementary differentiation the time derivative of V along the solutions of (1.5) can be estimated as follows

$$\begin{aligned}\dot{V} &= z^2 + xzg(x(t-\tau(t)),...,u(t-\tau(t))) + x^2f(x(t-\tau(t)),...,u(t-\tau(t))) \\ &= [z + 2^{-1}xg(x(t-\tau(t)),...,u(t-\tau(t)))]^2 \\ &\quad + \left[f(x(t-\tau(t)),...,u(t-\tau(t))) - \frac{1}{4}g^2(x(t-\tau(t)),...,u(t-\tau(t)))\right]x^2 \\ &\geq \left[f(x(t-\tau(t)),...,u(t-\tau(t))) - \frac{1}{4}g^2(x(t-\tau(t)),...,u(t-\tau(t)))\right]x^2 > 0.\end{aligned}$$

Thus V satisfies the property (K_2), (see [6]).

Further, it follows that $\dot{V} = 0 \Leftrightarrow x = 0$. In turn, this implies that

$$x = y = z = u = 0.$$

Thus V satisfies the property (K_3), (see [6]). This completes the proof of Theorem 2.1. □

Example 2.1. Consider nonlinear differential equation of fourth order with a variable deviating argument, $\tau(t) = t/2$:

$$\begin{aligned}x^{(4)} &+ x''' + \{2 + \frac{2}{1 + x^2(t/2) + ... + x'''^2(t/2)}\}x'' \\ &+ 4xx' + (9 + x^2(t/2) + ... + x'''^2(t/2))x = 0\end{aligned}$$

so that

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= u, \\ u' &= -u - \{2 + \frac{2}{1 + x^2(t/2) + ... + u^2(t/2)}\}z - 4xy \\ &\quad -\{9 + x^2(t/2) + ... + u^2(t/2)\}x(t) = 0. \end{aligned}$$

We have the following estimates:

$$a_1 = 1,$$
$$\tau(t) = t/2,$$
$$g(x(t-\tau(t)), ..., u(t-\tau(t))) = 2 + \frac{2}{1 + x^2(t/2) + ... + u^2(t/2)},$$
$$h(x) = 4x$$

and

$$f(x(t-\tau(t)), ..., u(t-\tau(t))) = 9 + x^2(t/2) + ... + u^2(t/2)$$

so that

$$\begin{aligned} f(.) - \frac{1}{4}g^2(.) &= 9 + x^2(t/2) + ... + u^2(t/2) \\ &\quad - \left[1 + \frac{1}{1 + x^2(t/2) + ... + u^2(t/2)}\right]^2 > 0. \end{aligned}$$

This shows that the zero solution of the above equation is unstable.

The second main result is the following theorem.

Theorem 2.2. *Suppose that*

$$p(0, z) = 0, \quad a_4 > 0 \quad \text{and } a_4 - \frac{1}{4}q^2(x(t-\tau(t)), ..., u(t-\tau(t))) > 0$$

for arbitrary $x(t-\tau(t)), ..., u(t-\tau(t))$, *and* $\frac{\partial p}{\partial z}(u, z) sgn u \leq 0$ *for arbitrary* u, z. *Then the zero solution of Eq. (1.4) is unstable for arbitrary* a_3.

Proof. Consider the Lyapunov function $V_1 = V_1(x, y, z, u)$ defined by

$$V_1 = -\int_0^u p(s, z)ds - a_3yu + \frac{1}{2}a_3z^2 - a_4xu + a_4yz$$

so that

$$V_1(0, \varepsilon^2, \varepsilon, 0) = a_4\varepsilon^3 + \frac{1}{2}a_3\varepsilon^4 > 0, (a_4 > 0), (a_3 \in \Re),$$

for sufficiently small ε. Indeed, if ε is an arbitrary positive constant, then

$$V_1(0,\varepsilon^2,\varepsilon,0)>0$$

for sufficiently small ε. Thus V_1 satisfies the property (K_1), (see [6]).

The time derivativeof V_1 along the solutions of (1.6) can be calculated as follows:

$$\begin{aligned}\dot{V}_1 &= -u'\,\{p(u,z)+a_3y+a_4x\}+a_4z^2-u\int_0^u \frac{\partial p}{\partial z}(s,z)ds\\ &= u'\,\{u'+q(x(t-\tau(t)),...,u(t-\tau(t)))z\}\\ &\quad +a_4z^2-u\int_0^u \frac{\partial p}{\partial z}(s,z)ds.\end{aligned}$$

The last estimate leads

$$\begin{aligned}\dot{V}_1 &= \{u'+2^{-1}q(x(t-\tau(t)),...,u(t-\tau(t)))z\}^2\\ &\quad +\{a_4-4^{-1}q^2(x(t-\tau(t)),...,u(t-\tau(t)))\}z^2\\ &\quad -u\int_0^u \frac{\partial p}{\partial z}(s,z)ds\end{aligned}$$

so that

$$\begin{aligned}\dot{V}_1 &\geq \{u'+2^{-1}q(x(t-\tau(t)),...,u(t-\tau(t)))z\}^2\\ &\quad +\{a_4-4^{-1}q^2(x(t-\tau(t)),...,u(t-\tau(t)))\}z^2>0.\end{aligned}$$

Thus V_1 satisfies the property (K_2), (see [6]).

On the other hand, $\dot{V}_1=0 \Leftrightarrow z=0$, this implies that $z=u=0$. System (1.6) and $\dot{V}_1=0$ leads that

$$a_3y+a_4x=0\Rightarrow a_3x'+a_4x=0.$$

Because of $x''=0$, it follows that x' =constant so that $a_3x'+a_4x=0\Rightarrow$ x =constant. However, this implies $x'=0$ since $a_4\neq 0$. Hence $a_4>0$ implies $x=0$. Thus V_1 satisfies the property (K_3), (see [6]). This completes the proof of Theorem 2.2. □

Example 2.2. Consider nonlinear differential equation of fourth order with a variable deviating argument, $\tau(t)=t/2$:

$$x^{(4)}-(arctgx'')x'''+2\cos(x(t/2)+...+x'''(t/2))x''+3x'+4x=0.$$

so that

$$\begin{aligned}x' &= y,\\ y' &= z,\\ z' &= u,\\ u' &= (arctgz)u-2\cos(x(t/2)+...+u(t/2))z-3y-4x.\end{aligned}$$

We have the following estimates:

$$\tau(t) = t/2, \qquad a_3 = 3, \qquad a_4 = 4,$$
$$p(u,z) = -\,(arctgz)u,$$
$$\frac{\partial p}{\partial z}(u,z)sgnu = -\frac{u}{1+z^2}sgnu \le 0,$$
$$q(x(t-\tau(t)),...,u(t-\tau(t))) = 2\cos(x(t/2)+...+u(t/2)),$$

so that

$$a_4 - \frac{1}{4}q^2(.) = 4 - \cos^2(x(t/2)+...+u(t/2)) > 0.$$

This shows that the zero solution of the above equation is unstable.

Acknowledgement

The author would like to express his sincere appreciation to the reviewer for his/her helpful comments, corrections and suggestions which helped with improving the presentation and quality of this work.

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DOI: 10.7862/rf.2013.10

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Received 1.08.2012, *Revisted 29.08.2012*, *Accepted 25.10.2013*