Supra $b$-compact and supra $b$-Lindelöf spaces

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Abstract: In this paper we introduce the notion of supra $b$-compact spaces and investigate its several properties and characterizations. Also we introduce and study the notion of supra $b$-Lindelöf spaces.

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1. Introduction and preliminaries

In 1983, A. S. Mashhour et al. [3] introduced the supra topological spaces. In 1996, D. Andrijevic [1] introduced and studied a class of generalized open sets in a topological space called $b$-open sets. This type of sets discussed by El-Atike [2] under the name of $\gamma$-open sets. In 2010, O. R. Sayed et al. [4] introduced and studied a class of sets and maps between topological spaces called supra $b$-open sets and supra $b$-continuous functions respectively. Now we introduce the concepts of supra $b$-compact and supra $b$-Lindelöf spaces and investigate several properties for these concepts.

Throughout this paper $(X, \tau)$, $(Y, \rho)$ and $(Z, \sigma)$ (or simply $X$, $Y$ and $Z$) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of $(X, \tau)$, the closure and the interior of $A$ in $X$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. The complement of $A$ is denoted by $X - A$. In the space $(X, \tau)$, a subset $A$ is said to be $b$-open [1] if $A \subseteq \text{Cl}(\text{Int}(A)) \cup \text{Int}(	ext{Cl}(A))$. The family of all $b$-open sets of $(X, \tau)$ is denoted by $\text{BO}(X)$. A subcollection $\mu \subseteq 2^X$ is called a supra topology [3] on $X$ if $X \in \mu$ and $\mu$ is closed under arbitrary union. $(X, \mu)$ is called a supra topological space. The elements of $\mu$ are said to be supra open in $(X, \mu)$ and the complement of a supra open set is called a supra closed set. The supra closure of a set $A$, denoted by $\text{Cl}^\mu(A)$, is the intersection of all supra closed sets including $A$. The supra interior of a set $A$, denoted by $\text{Int}^\mu(A)$, is the union of all supra open sets included in $A$. The supra topology $\mu$ on $X$ is associated with the topology $\tau$ if $\tau \subseteq \mu$. 
Definition 1.1 [4] Let \((X, \mu)\) be a supra topological space. A set \(A\) is called a supra \(b\)-open set if \(A \subseteq \text{Cl}^\mu(\text{Int}^\mu(A)) \cup \text{Int}^\mu(\text{Cl}^\mu(A))\). The complement of a supra \(b\)-open set is called a supra \(b\)-closed set.

Theorem 1.2 [4]. (i) Arbitrary union of supra \(b\)-open sets is always supra \(b\)-open.

(ii) Finite intersection of supra \(b\)-open sets may fail to be supra \(b\)-open.

Definition 1.3 [4] The supra \(b\)-closure of a set \(A\), denoted by \(\text{Cl}^\mu_b(A)\), is the intersection of supra \(b\)-closed sets including \(A\). The supra \(b\)-interior of a set \(A\), denoted by \(\text{Int}^\mu_b(A)\), is the union of supra \(b\)-open sets included in \(A\).

2. Supra \(b\)-compact and supra \(b\)-Lindelöf spaces

Definition 2.1 A collection \(\{U_\alpha : \alpha \in \Delta\}\) of supra \(b\)-open sets in a supra topological space \((X, \mu)\) is called a supra \(b\)-open cover of a subset \(B\) of \(X\) if \(B \subseteq \bigcup \{U_\alpha : \alpha \in \Delta\}\).

Definition 2.2 A supra topological space \((X, \mu)\) is called supra \(b\)-compact (resp. supra \(b\)-Lindelöf) if every supra \(b\)-open cover of \(X\) has a finite (resp. countable) subcover.

The proof of the following theorem is straightforward and thus omitted.

Theorem 2.3 If \(X\) is finite (resp. countable) then \((X, \mu)\) is supra \(b\)-compact (resp. supra \(b\)-Lindelöf) for any supra topology \(\mu\) on \(X\).

Definition 2.4 A subset \(B\) of a supra topological space \((X, \mu)\) is said to be supra \(b\)-compact (resp. supra \(b\)-Lindelöf) relative to \(X\) if, for every collection \(\{U_\alpha : \alpha \in \Delta\}\) of supra \(b\)-open subsets of \(X\) such that \(B \subseteq \bigcup \{U_\alpha : \alpha \in \Delta\}\), there exists a finite (resp. countable) subset \(\Delta_0\) of \(\Delta\) such that \(B \subseteq \bigcup \{U_\alpha : \alpha \in \Delta_0\}\).

Notice that if \((X, \mu)\) is a supra topological space and \(A \subseteq X\) then \(\mu_A = \{U \cap A : U \in \mu\}\) is a supra topology on \(A\).

\((A, \mu_A)\) is called a supra subspace of \((X, \mu)\).

Definition 2.5 A subset \(B\) of a supra topological space \((X, \mu)\) is said to be supra \(b\)-compact (resp. supra \(b\)-Lindelöf) if \(B\) is supra \(b\)-compact (resp. supra \(b\)-Lindelöf) as a supra subspace of \(X\).

Theorem 2.6 Every supra \(b\)-closed subset of a supra \(b\)-compact space \(X\) is supra \(b\)-compact relative to \(X\).

Prof: Let \(A\) be a supra \(b\)-closed subset of \(X\) and \(\bar{U}\) be a cover of \(A\) by supra \(b\)-open subsets of \(X\). Then \(\bar{U}^* = \bar{U} \cup \{X - A\}\) is a supra \(b\)-open cover of \(X\). Since \(X\) is supra \(b\)-compact, \(\bar{U}^*\) has a finite subcover \(\bar{U}^{**}\) for \(X\). Now \(\bar{U}^{**} - \{X - A\}\) is a finite subcover of \(\bar{U}\) for \(A\), so \(A\) is supra \(b\)-compact relative to \(X\). □
Theorem 2.7 Every supra $b$-closed subset of a supra $b$-Lindelöf space $X$ is supra $b$-Lindelöf relative to $X$.

Prof: Similar to the proof of the above theorem. ■

Theorem 2.8 Every supra subspace of a supra topological space $(X,\mu)$ is supra $b$-compact relative to $X$ if and only if every supra $b$-open subspace of $X$ is supra $b$-compact relative to $X$.

Prof: $\Rightarrow$ Is clear.

$\Leftarrow$ Let $Y$ be a supra subspace of $X$ and let $\bar{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $Y$ by supra $b$-open sets in $X$. Now let $V = \bigcup \bar{U}$, then $V$ is a supra $b$-open subset of $X$, so it is supra $b$-compact relative to $X$. But $\bar{U}$ is a cover of $V$ so $\bar{U}$ has a finite subcover $\bar{U}^*$ for $V$. Then $V \subseteq \bigcup \bar{U}^*$ and therefore $V \subseteq \bigcup \bar{U}^*$. So $\bar{U}^*$ is a finite subcover of $\bar{U}$ for $Y$. Then $Y$ is supra $b$-compact relative to $X$. ■

Theorem 2.9 Every supra subspace of a supra topological space $(X,\mu)$ is supra $b$-Lindelöf relative to $X$ if and only if every supra $b$-open subspace of $X$ is supra $b$-Lindelöf relative to $X$.

Prof: Similar to the proof of the above theorem. ■

For a family $\tilde{A}$ of subsets of $X$, if all finite intersection of the elements of $\tilde{A}$ are non-empty, we say that $\tilde{A}$ has the finite intersection property.

Theorem 2.10 A supra topological space $(X,\mu)$ is supra $b$-compact if and only if every supra $b$-closed family of subsets of $X$ having the finite intersection property, has a non-empty intersection.

Prof: $\Rightarrow$ Let $\tilde{A} = \{A_\alpha : \alpha \in \Delta\}$ be a supra $b$-closed family of subsets of $X$ which has the finite intersection property. Suppose that $\cap \{A_\alpha : \alpha \in \Delta\} \neq \phi$. Let $\bar{U} = \{X - A_\alpha : \alpha \in \Delta\}$ then $\bar{U}$ is a supra $b$-open cover of $X$. Then $\bar{U}$ has a finite subcover $\bar{U}^* = \{X - A_{\alpha_1}, X - A_{\alpha_2}, ..., X - A_{\alpha_n}\}$. Now $\tilde{A} = \{A_{\alpha_1}, A_{\alpha_2}, ..., A_{\alpha_n}\}$ is a finite subfamily of $\tilde{A}$ with $\cap \{A_{\alpha_i} : i = 1, 2, ..., n\} = \phi$ which is a contradiction.

$\Leftarrow$ Let $\bar{U} = \{U_\alpha : \alpha \in \Delta\}$ be a supra $b$-open cover of $X$. Suppose that $\tilde{U}$ has no finite subcover. Now $\tilde{A} = \{X - U_\alpha : \alpha \in \Delta\}$ is a supra $b$-closed family of subsets of $X$ which has the finite intersection property. So by assumption we have $\cap \{X - U_\alpha : \alpha \in \Delta\} \neq \phi$. Then $\cup \{U_\alpha : \alpha \in \Delta\} \neq X$ which is a contradiction. ■

The proof of the following theorem is straightforward and thus omitted.

Theorem 2.11 The finite (resp. countable) union of supra $b$-compact (resp. supra $b$-Lindelöf) sets relative to a supra topological space $X$ is supra $b$-compact (resp. supra $b$-Lindelöf) relative to $X$.

Theorem 2.12 Let $A$ be a supra $b$-compact (resp. supra $b$-Lindelöf) set relative to a supra topological space $X$ and $B$ be a supra $b$-closed subset of $X$. Then $A \cap B$ is supra $b$-compact (resp. supra $b$-Lindelöf) relative to $X$. 
**Definition 2.13** [4] Let $(X, \tau)$ and $(Y, \rho)$ be two topological spaces and $\mu$ be an associated supra topology with $\tau$. A function $f : (X, \tau) \rightarrow (Y, \rho)$ is called a supra $b$-continuous function if the inverse image of each supra open set in $Y$ is a supra $b$-open set in $X$.

**Theorem 2.14** A supra $b$-continuous image of a supra $b$-compact space is compact.

**Prof:** Similar to the proof of the above theorem.

**Definition 2.16** Let $(X, \tau)$ and $(Y, \rho)$ be two topological spaces and $\mu, \eta$ be associated supra topologies with $\tau$ and $\rho$ respectively. A function $f : (X, \tau) \rightarrow (Y, \rho)$ is called a supra $b$-irresolute function if the inverse image of each supra $b$-open set in $Y$ is a supra $b$-open set in $X$.

**Theorem 2.17** If a function $f : X \rightarrow Y$ is supra $b$-irresolute and a subset $B$ of $X$ is supra $b$-compact relative to $X$, then $f(B)$ is supra $b$-compact relative to $Y$.

**Prof:** Similar to the proof of the above theorem.

**Definition 2.19** [4]. A function $f : (X, \tau) \rightarrow (Y, \rho)$ is called a supra $b$-open function if the image of each open set in $X$ is a supra $b$-open set in $(Y, \eta)$.

The proof of the following theorem is straightforward and thus omitted.
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Theorem 2.20 Let $f : (X, \tau) \to (Y, \rho)$ be a supra $b$-open surjection and $\eta$ be a supra topology associated with $\rho$. If $(Y, \eta)$ is supra $b$-compact (resp. supra $b$-Lindelöf) then $(X, \tau)$ is compact (resp. Lindelöf).

Definition 2.21 A subset $F$ of a supra topological space $(X, \mu)$ is called supra $b$-$F_\sigma$-set if $F = \bigcup \{F_i : i = 1, 2, \ldots\}$ where $F_i$ is a supra $b$-closed subset of $X$ for each $i = 1, 2, \ldots$.

Theorem 2.22 A supra $b$-$F_\sigma$-set $F$ of a supra $b$-Lindelöf space $X$ is supra $b$-Lindelöf relative to $X$.

Prof: Let $F = \bigcup \{F_i : i = 1, 2, \ldots\}$ where $F_i$ is a supra $b$-closed subset of $X$ for each $i = 1, 2, \ldots$. Let $\hat{U}$ be a cover of $F$ by supra $b$-open sets in $X$, then $\hat{U}$ is a cover of $F_i$ for each $i = 1, 2, \ldots$ by supra $b$-open subsets of $X$. Since $F_i$ is supra $b$-Lindelöf relative to $X$, $\hat{U}$ has a countable subcover $\hat{U}_i = \{U_{i1}, U_{i2}, \ldots\}$ for $F_i$ for each $i = 1, 2, \ldots$. Now $\bigcup \{\hat{U}_i : i = 1, 2, \ldots\}$ is a countable subcover of $\hat{U}$ for $F$. So $F$ is supra $b$-Lindelöf relative to $X$. 

References


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