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Convolution properties of subclasses of analytic functions associated with the Dziok-Srivastava operator

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Abstract: The aim of this paper is to introduce two new classes of analytic function by using principle of subordination and the Dziok-Srivastava operator. We further investigate convolution properties for these calsses. We also find necessary and sufficient condition and coefficient estimate for them.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in C : |z| < 1\}$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)(0 \leq \alpha < 1)$ denote the subclasses of $\mathcal{A}$ that consists, respectively, of starlike of order α and convex of order α in the disk $\mathbb{U}$. It is well known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(\alpha) = \mathcal{K}(0) = \mathcal{K}$.

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$is subordinate to $g(z)$, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which by definition is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$, for all $z \in \mathbb{U}$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence :

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For the fucntion $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{1.2}$$

the Hadmard product or convolution of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \tag{1.3}$$

Making use of principle of subordination between analytic functions. We introduce the subclasses $\mathcal{S}^*[\lambda, \phi]$ and $\mathcal{K}[\lambda, \phi]$ of the class $\mathcal{A}$ for $-1 \leq \lambda \leq 1$ which are defined by

$$\mathcal{S}^*[\lambda, \phi] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{[(1-\lambda)f(z) + \lambda z f'(z)]} \prec \phi(z) \ \ (z \in \mathbb{U}) \right\} \tag{1.4}$$

and

$$\mathcal{K}[\lambda, \phi] = \left\{ f \in \mathcal{A} : \frac{zf''(z) + f'(z)}{[f'(z) + \lambda z f''(z)]} \prec \phi(z) \ \ (z \in \mathbb{U}) \right\} \tag{1.5}$$

For complex parameters $a_1, ..., a_q; b_1, ..., b_s (b_j \notin \mathcal{Z}_0^- = \{0, -1, -2, ...\}; j = 1, ..., s)$, we define the generalized hypergeometric function ${}_qF_s(a_1, ..., a_i, ..., a_q; b_1, ..., b_s; z)$ by [12] the following infinite series:

$${}_qF_s(a_1, ..., a_i, ..., a_q; b_1, ..., b_s; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k ... (a_q)_k}{(b_1)_k ... (b_s)_k} \frac{z^k}{k!} \tag{1.6}$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where $(\alpha)_k$ is Pochhammer symbol defined by

$$(\alpha)_k = \begin{cases} 1 & for \ \ k = 0 \\ \alpha(\alpha+1)...(\alpha+k-1) & for \ k \in \mathbb{N} \end{cases}$$

Dziok and Srivastava [4] considered a linear operator $H(a_1, ..., a_q; b_1, ..., b_s) : \mathcal{A} \to \mathcal{A}$ defined by the following Hadamard product:

$$H(a_1, ..., a_q; b_1, ..., b_s)f(z) = h(a_1, ..., a_q; b_1, ..., b_s; z) * f(z) \tag{1.7}$$

where

$$h(a_1, ..., a_i, ..., a_q; b_1, ..., b_s; z) = z {}_qF_s(a_1, ..., a_q; b_1, ..., b_s; z)$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0; z \in \mathbb{U}).$$

If $f(z) \in \mathcal{A}$ is given by (1.1), then we have

$$H(a_1, ..., a_q; b_1, ..., b_s)f(z) = z + \sum_{k=2}^{\infty} \Gamma_k[a_1; b_1] a_k z^k, \tag{1.8}$$

where

$$\Gamma_k[a_1;b_1] = \frac{(a_1)_{k-1}...(a_q)_{k-1}}{(b_1)_{k-1}...(b_s)_{k-1}(k-1)!} \tag{1.9}$$

The Dziok-Srivastava linear operator $H_{q,s}[a_1;b_1]$ includes various other operators, which were considered in earlier works. We can quote here for example linear operators introduced by Carlson and Shaffer, Bernardi, Libera and Livingston, Choi, Saigo and Srivastava, Kim and Srivastava, Srivastava and Owa, Cho, Kwon and Srivastava, Ruscheweyh, Hohlov, Salagean, Noor, and others (see for details [8], [9] and []).

In recent years, many interesting subclasses of analytic functions associated with the Dziok-Srivastava operator $H_{q,s}[a_1;b_1]$ and its many special cases were investigated by, for example, Murugusundaramoorthy and Magesh [7], Srivastava et al. ([13],[14]),Wang et al. [15] and others.

In this paper, we investigate convolution properties of the classes $\mathcal{S}^*[a_1;\lambda,\phi]$ and $\mathcal{K}[a_1;\lambda,\phi]$ associated with the operator $H_{q,s}[a_1;b_1]$. Using convolution properties, we find the necessary and sufficient condition and coefficient estimate for these classes.

2. Convolution properties

We assume that $0<\theta<2\pi, -1\leq\lambda\leq 1$ throughout this section and $\Gamma_k[a_1;b_1]$is defined by (1.9)

Theorem 1. *The function $f(z)$ defined by (1.1) is in the class $\mathcal{S}^*[\lambda,\phi]$ if and only if.*

$$\frac{1}{z}\left[f(z)*\frac{z-\frac{(\lambda-1)\phi(e^{i\theta})}{1-\phi(e^{i\theta})}z^2}{(1-z)^2}\right]\neq 0 \quad (z\in\mathbb{U}, 0<\theta 2\pi) \tag{2.1}$$

Proof. A function $f(z)$ is in the class $\mathcal{S}^*[\lambda,\phi]$ if and only if

$$\frac{zf'(z)}{[(1-\lambda)f(z)+\lambda zf'(z)]}\neq\phi(e^{i\theta})\ (z\in\mathbb{U},\ 0<\theta<2\pi) \tag{2.2}$$

which is equivalent to

$$zf'(z)\neq\phi(e^{i\theta})\left[(1-\lambda)f(z)+\lambda zf'(z)\right],$$

$$\frac{1}{z}\left[zf'(z)\left[1-\lambda\phi(e^{i\theta})\right]-(1-\lambda)\phi(e^{i\theta})f(z)\right]\neq 0. \tag{2.3}$$

Since
$f(z)=f(z)*\frac{1}{(1-z)}$ and $zf'(z)=f(z)*\frac{1}{(1-z)^2}$,
The equation (2.3) can be written as

$$\frac{1}{z}\left[f(z)*\left((1-\lambda\phi(e^{i\theta}))\frac{z}{(1-z)^2}-(1-\lambda)\phi(e^{i\theta})\frac{z}{1-z}\right)\right]$$

$$=\frac{1-\phi(e^{i\theta})}{z}\left[f(z)*\frac{z-((\lambda-1)\phi(e^{i\theta})/(1-\phi(e^{i\theta})))z^2}{(1-z)^2}\right]\neq 0,(0<\theta<2\pi). \tag{2.4}$$

this completes the proof of Theorem 1.

Theorem 2. *The function $f(z)$ defined by (1.1) is in the class $\mathcal{K}[\lambda,\phi]$ if and only if.*

$$\frac{1}{z}\left[f(z)*\frac{z-((1+(1-2\lambda)\phi(e^{i\theta}))/(\phi(e^{i\theta})-1)z^2}{(1-z)^3}\right]\neq 0,(z\in\mathbb{U}). \tag{2.5}$$

Proof. Let us take

$$g(z)=\frac{z-((\lambda-1)\phi(e^{i\theta})/(1-\phi(e^{i\theta})))z^2}{(1-z)^2}, \tag{2.6}$$

from which we get

$$zg'(z)=\frac{z-((1+(1-2\lambda)\phi(e^{i\theta}))/(\phi(e^{i\theta})-1))z^2}{(1-z)^3}\ (0<\theta<2\pi). \tag{2.7}$$

Also from the identity $zf'(z)*g(z)=f(z)*zg'(z),(f,g\in\mathcal{A})$ and the fact that

$$f(z)\in\mathcal{K}[\lambda,\phi]\Longleftrightarrow zf'(z)\in\mathcal{S}^*[\lambda,\phi].$$

the result (2.5) follows from Theorem 1.

Theorem 3. *A necessary and sufficient condition for the function $f(z)$ defined by (1.1) to be in the class $\mathcal{S}^*_{q,s}[a_1;\lambda,\phi]$ is that.*

$$1+\sum_{k=2}^{\infty}\frac{(1-\lambda)\phi(e^{i\theta})+k(\lambda\phi(e^{i\theta})-1)}{\phi(e^{i\theta})-1}\Gamma_k[a_1,b_1]a_kz^{k-1}\neq 0\ \ (z\in\mathbb{U},0<\theta<2\pi) \tag{2.8}$$

Proof. From Theorem 1, we can say that $f(z)\in S^*_{q,s}[a_1,\lambda,\phi]$ if and only if

$$\frac{1}{z}\left[H_{q,s}[a_1,b_1]f(z)*\frac{z-((\lambda-1)\phi(e^{i\theta})/(1-\phi(e^{i\theta})))z^2}{(1-z)^2}\right]\neq 0,\ \ (z\in\mathbb{U},0<\theta<2\pi). \tag{2.9}$$

From (1.8), the left hand side of (2.9) can be written as

$$\frac{1}{z}\left[H_{q,s}[a_1,b_1]f(z)*\left(\frac{z}{(1-z^2)}-\frac{(1-\lambda)\phi(e^{i\theta})}{\phi(e^{i\theta})-1}\frac{z^2}{(1-z)^2}\right)\right],(0<\theta<2\pi). \tag{2.10}$$

$$=\frac{1}{z}\left[z(H_{q,s}(a_1,b_1))f(z)' - \frac{(1-\lambda)\phi(e^{i\theta})}{\phi(e^{i\theta})-1}\left\{z(H_{q,s}(a_1,b_1))f(z)'-(H_{q,s}(a_1,b_1))f(z)\right\}\right]. \tag{2.11}$$

$$=1+\sum_{k=2}^{\infty}\frac{(1-\lambda)\phi(e^{i\theta})+k(\lambda\phi(e^{i\theta})-1)}{\phi(e^{i\theta})-1}\Gamma_k[a_1,b_1]a_kz^{k-1},\ (0<\theta<2\pi). \tag{2.12}$$

Thus the proof is completed.

Theorem 4. *A necessary and sufficient condition for the function $f(z)$ defined by (1.1) to be in the class $\mathcal{K}_{q,s}[a_1;\lambda,\phi]$ is that*

$$1+\sum_{k=2}^{\infty} k\frac{(1-\lambda)\phi(e^{i\theta})+k(\lambda\phi(e^{i\theta})-1)}{\phi(e^{i\theta})-1}\Gamma_k[a_1,b_1]a_k z^{k-1}\neq 0,\ \ (z\in\mathbb{U}, 0<\theta<2\pi) \tag{2.13}$$

Proof. From Theorem 1, we find that $f(z)\in\mathcal{K}_{q,s}[a_1;\lambda,\phi]$ if and only if

$$\frac{1}{z}\left[H_{q,s}[a_1,b_1]f(z)*\frac{z-((1+(1-2\lambda)\phi(e^{i\theta}))/(\phi(e^{i\theta})-1)z^2}{(1-z)^3}\right]\neq 0, (z\in\mathbb{U}). \tag{2.14}$$

Using the definition (1.8), the above equation can be written as

$$\begin{aligned}
&\frac{1}{z}\left[H_{q,s}[a_1,b_1]f(z)*\left(\frac{z}{(1-z)^3}-\frac{(1+(1-2\lambda)\phi(e^{i\theta}))}{(\phi(e^{i\theta})-1)}\frac{z}{(1-z)^3}\right)\right]\\
&=\frac{1}{z}\left[\frac{z}{2}\left(zH_{q,s}[a_1,b_1]f(z)\right)''-\frac{(1+(1-2\lambda)\phi(e^{i\theta}))}{2(\phi(e^{i\theta})-1)}z^2(H_{q,s}[a_1,b_1]f(z))''\right]\\
&=1+\sum_{k=2}^{\infty}k\frac{(1-\lambda)\phi(e^{i\theta})+k(\lambda\phi(e^{i\theta})-1)}{\phi(e^{i\theta})-1}\Gamma_k[a_1,b_1]a_k z^{k-1}
\end{aligned} \tag{2.15}$$

which proves the Theorem.

Theorem 5. *If the function $f(z)$ defined by (1.1) belongs to $\mathcal{S}^*_{q,s}[a_1;\lambda,\phi]$ then*

$$\sum_{k=2}^{\infty}(1-\lambda)|\phi(e^{i\theta})|-|(\lambda\phi(e^{i\theta})-1)|k)\Gamma_k[a_1,b_1]|a_k|\leq|1-\phi(e^{i\theta})| \tag{2.16}$$

Proof. Since

$$\begin{aligned}
&\left|1-\sum_{k=2}^{\infty}\frac{(1-\lambda)\phi(e^{i\theta})+k(\lambda\phi(e^{i\theta})-1)}{1-\phi(e^{i\theta})}\Gamma_k[a_1,b_1]a_k z^{k-1}\right| \qquad (z\in\mathbb{U})\\
&\geq 1-\sum_{k=2}^{\infty}\left|\frac{(1-\lambda)\phi(e^{i\theta})+k(\lambda\phi(e^{i\theta})-1)}{1-\phi(e^{i\theta})}\right|\Gamma_k[a_1,b_1]\,|a_k|\\
&=>\sum_{k=2}^{\infty}\left((1-\lambda)\left|\phi(e^{i\theta})\right|-\left|(\lambda\phi(e^{i\theta})-1)\right|k\right)\Gamma_k[a_1,b_1]\,|a_k|\leq\left|1-\phi(e^{i\theta})\right|
\end{aligned}$$

Theorem 6. *If the function $f(z)$ defined by (1.1) belongs to $\mathcal{K}_{q,s}[a_1;\lambda,\phi]$ then*

$$\sum_{k=2}^{\infty}((1-\lambda)|\phi(e^{i\theta})|-|(\lambda\phi(e^{i\theta})-1)|k)k\Gamma_k[a_1,b_1]|a_k|\leq|1-\phi(e^{i\theta})| \tag{2.17}$$

Remark. Putting $\phi(e^{i\theta})=\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}$ and $\lambda=0$ in theorems 1 to 6, we get the results given recently by Aouf and Seoudy [2], Some of the results by Aouf and Seoudy also

contain the result due to Silverman ([10], [11]) and Ahuja [1].
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