Journal of
Mathematics
and Applications

JMA No 44, pp 71-73 (2021)

Weakly Locally Uniformly Rotund Norm which is not Locally Uniformly Rotund

Marek Malec

Abstract: The aim of this paper is to provide a proof of the fact that a weakly locally uniformly rotund norm does not have to be locally uniformly rotund. This result is well-known for experts in Geometry of Banach Spaces. However, since the justification of this result is omitted in the literature, we believe that the present note may be helpful for students or novices in the theory.

AMS Subject Classification: 46B20.
Keywords and Phrases: Locally uniformly rotund norm; Weakly locally uniformly rotund norm.

1. Introduction

The notion of locally uniformly rotund space was introduced by A. R. Lovaglia in [3] and widely studied afterwards. In our considerations we will accept the following definition of the concept of the (weak) local uniform rotundity (cf. [1]).

Definition 1.1. A normed space $(X, \|\cdot\|)$ is *locally uniformly rotund* (*weakly locally uniformly rotund*) if for $x \in X$, $(x_n) \subset X$, such that $\|x\| = 1$, $\|x_n\| = 1$ for $n \in \mathbb{N}$ and

$$\left\| \frac{x_n + x}{2} \right\| \to 1,$$

we get that x_n converges to x (x_n converges weakly to x).

Let us consider the space c_0 of the sequences convergent to zero with the norm given by the formula

$$\|\!|x|\!\| = \|x\|_\infty + \left(\sum_{k=1}^{\infty} 2^{-k}|x_k|^2\right)^{\frac{1}{2}} \tag{1.1}$$

for $x = (x_k) \in c_0$.

In [4] authors used c_0 with the norm (1.1) as an example of rotund norm, which is not locally uniformly rotund. Later, S. Draga proved in [2] that the considered norm is weakly locally uniformly rotund. It is worthwhile mentioning that there is no proof of the fact observed in [4] and to our best knowledge there is no such proof in other papers (a similar example in the space $C[0,1]$ together with a proof was established in [3], pp. 229-230). In this paper we fill that gap. Namely, we will show now that the norm (1.1) on the space c_0 is not locally uniformly rotund.

2. The proof

Let $x = \left(2-\sqrt{2}, 0, 0, ...\right) \in c_0$ ($\|\!|x|\!\| = 1$). Consider $(x_n) \subset c_0$, where

$$x_n = \left(\alpha_n, 0, 0, \dots, 0, \frac{1}{2}, 0, 0, ...\right)$$

and $\frac{1}{2}$ lies on $n+1$ coordinate, $\alpha_n > 0$. We want to choose α_n such that $\|\!|x_n|\!\| = 1$ for $n \in \mathbb{N}$. We are going to explain how to fulfill this requirement. We have

$$1 = \|\!|x_n|\!\| = \alpha_n + \sqrt{\frac{1}{2}\alpha_n^2 + \left(\frac{1}{2}\right)^{n+1} \cdot \left(\frac{1}{2}\right)^2},$$

$$\frac{1}{2}\alpha_n^2 - 2\alpha_n + 1 - \left(\frac{1}{2}\right)^{n+3} = 0.$$

Hence, we obtain

$$\alpha_n = 2 - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}} \qquad \text{or} \qquad \alpha_n = 2 + \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}}.$$

Since $1 - \alpha_n \geq 0$, we conclude that

$$x_n = \left(2 - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}}, 0, 0, \dots, 0, \frac{1}{2}, 0, 0, ...\right).$$

Hence, we get

$$\left\|\!\left|\frac{x_n + x}{2}\right|\!\right\| = \left\|\!\left|\left(\frac{2 - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}} + 2 - \sqrt{2}}{2}, 0, 0, \dots, 0, \frac{1}{4}, 0, 0, ...\right)\right|\!\right\|$$

$$= \frac{4 - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}} - \sqrt{2}}{2} + \sqrt{\frac{1}{2}\left(\frac{4 - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}} - \sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^{n+1}\left(\frac{1}{4}\right)^2}$$

$$= 2 - \frac{\sqrt{2} + \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}}}{2} + \sqrt{\frac{1}{2}\left(2 - \frac{\sqrt{2} + \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}}}{2}\right)^2 + \left(\frac{1}{2}\right)^{n+5}}.$$

Passing with n to infinity, we obtain

$$\left|\!\left|\!\left| \frac{x_n + x}{2} \right|\!\right|\!\right| \xrightarrow[n\to\infty]{} 2 - \sqrt{2} + \sqrt{\frac{1}{2}\left(2 - \sqrt{2}\right)} = 1.$$

On the other hand, we have

$$|\!|\!| x_n - x |\!|\!| \geq \|x_n - x\|_\infty = \left\| \left(\sqrt{2} - \sqrt{2 + \left(\frac{1}{2}\right)^{n+2}}, 0, 0, \dots, 0, \frac{1}{2}, 0, 0, \dots\right) \right\|_\infty \geq \frac{1}{2},$$

for all $n \in \mathbb{N}$. This shows that the sequence x_n does not converge to x in the norm $|\!|\!| \cdot |\!|\!|$. As a consequence, the norm $|\!|\!| \cdot |\!|\!|$ defined by (1.1) is not locally uniformly rotund. □

References

[1] J. Diestel, *Geometry of Banach Spaces - Selected Topics*, Lecture Notes in Mathematics 485, Springer-Verlag, Berlin Heidelberg, 1975.

[2] S. Draga, *On weakly locally uniformly rotund norms which are not locally uniformly rotund*, Colloquium Mathematicum 138 (2) (2015) 241–246.

[3] A.R. Lovaglia, *Locally uniformly convex Banach spaces*, Trans. Amer. Math. Soc. 78 (1955) 225–238.

[4] A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia, *A Nonlinear Transfer Technique for Renorming*, Lecture Notes in Mathematics 1951, Springer, Berlin, 2009.

DOI: 10.7862/rf.2021.5

Marek Malec
email: `marekmalec0@gmail.com`
ORCID: 0000-0001-9892-1403
Faculty of Mathematics and Applied Physics
Rzeszów University of Technology
al. Powstańców Warszawy 8,
35-959 Rzeszów
POLAND

Received 05.08.2021 *Accepted 23.08.2021*