Preserving subordination and superordination
results of generalized
Srivastava-Attiya operator

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ABSTRACT: In this paper, we obtain some subordination and superordination-preserving results of the generalized Srivastava-Attiya operator. Sandwich-type result is also obtained.

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1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_nz^n + a_{n+1}z^{n+1} + ...$, with $H_0 = H[0, 1]$ and $H = H[1, 1]$. Denote $A(p)$ by the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}; z \in U) \quad (1.1)$$

and let $A(1) = A$. For $f, F \in H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in U)$, such that $f(z) = F(\omega(z))$. In such a case we write $f(z) \prec F(z)$. If $F$ is univalent, then $f(z) \sim F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [14] and [15]).

Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ and satisfies the first order differential subordination:

$$\phi \left(p(z), zp'(z); z\right) \prec h(z), \quad (1.2)$$

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then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2) if $p(z) < q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} < q$ for all dominants of (1.2) is called the best dominant. If $p(z)$ and $\phi \left(p(z), zp'(z); z\right)$ are univalent in $U$ and if $p(z)$ satisfies the first order differential superordination:

$$h(z) < \phi \left(p(z), zp'(z); z\right),$$  \hspace{1cm} (1.3)

then $p(z)$ is a solution of the differential superordination (1.3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) < p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $\tilde{q} < q$ for all subordinants of (1.3) is called the best subordinant (see [14] and [15]).

The general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ is defined by:

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s},$$ \hspace{1cm} (1.4)

$(a \in \mathbb{C}\backslash\mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0, -1, -2, ...\}; s \in \mathbb{C}$ when $|z| < 1; R\{s\} > 1$ when $|z| = 1)$.

For interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ (see [3], [8], [9], [11] and [19]).

Recently, Srivastava and Attiya [18] introduced the linear operator $L_{s,b} : A \to A$, defined in terms of the Hadamard product by

$$L_{s,b}(f)(z) = G_{s,b}(z) * f(z) \ (z \in U; b \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}),$$ \hspace{1cm} (1.5)

where for convenience,

$$G_{s,b} = (1 + b)^s[\Phi(z, s, b) - b^{-s}] \ (z \in U).$$ \hspace{1cm} (1.6)

The Srivastava-Attiya operator $L_{s,b}$ contains among its special cases, the integral operators introduced and investigated by Alexander [1], Libera [7] and Jung et al. [6].

Analogous to $L_{s,b}$, Liu [10] defined the operator $J_{p,s,b} : A(p) \to A(p)$ by

$$J_{p,s,b}(f)(z) = G_{p,s,b}(z) * f(z) \ (z \in U; b \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}; p \in \mathbb{N}),$$ \hspace{1cm} (1.7)

where

$$G_{p,s,b} = (1 + b)^s[\Phi_p(z, s, b) - b^{-s}]$$

and

$$\Phi_p(z, s, b) = \frac{1}{b^p} + \sum_{n=0}^{\infty} \frac{z^{n+p}}{(n + 1 + b)^s}. \hspace{1cm} (1.8)$$

It is easy to observe from (1.7) and (1.8) that

$$J_{p,s,b}(f)(z) = z^p + \sum_{n=1}^{\infty} \left( \frac{1 + b}{n + 1 + b} \right)^s a_{n+p} z^{n+p}.$$ \hspace{1cm} (1.9)
We note that
(i) \( J_{p,0,b}(f)(z) = f(z) \);
(ii) \( J_{1,1,0} (f)(z) = L f(z) = \int_0^z \frac{f(t)}{t} dt \), where the operator \( L \) was introduced by Alexander [11];
(iii) \( J_{s,b,b}(f)(z) = L_{s,b} f(z) \) (\( s, b \in \mathbb{C}, b \in \mathbb{C} \setminus \mathbb{Z}_0^+ \)), where the operator \( L_{s,b} \) was introduced by Srivastava and Attiya [18];
(iv) \( J_{p,1,\nu+p-1}(f)(z) = F_{\nu,p}(f(z)) \) (\( \nu > -p, p \in \mathbb{N} \)), where the operator \( F_{\nu,p} \) was introduced by Choi et al. [4];
(v) \( J_{p,0,p}(f)(z) = I_p^\alpha f(z) \) (\( \alpha \geq 0, p \in \mathbb{N} \)), where the operator \( I_p^\alpha \) was introduced by Shams et al. [17];
(vi) \( J_{p,m,p-1}(f)(z) = J_p^m f(z) \) (\( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N} \)), where the operator \( J_p^m \) was introduced by El-Ashwah and Aouf [5];
(vii) \( J_{p,m,p+1-1}(f)(z) = J_p^m (l) f(z) \) (\( m \in \mathbb{N}_0, p \in \mathbb{N}, l \geq 0 \)), where the operator \( J_p^m (l) \) was introduced by El-Ashwah and Aouf [5].

It follows from (1.9) that:

\[
z (J_{p,s+1,b}(f)(z))^\prime = (b + 1) J_{p,s,b}(f)(z) - (b + 1 - p) J_{p,s+1,b}(f)(z).
\]

To prove our results, we need the following definitions and lemmas.

**Definition 1** [14]. Denote by \( F \) the set of all functions \( q(z) \) that are analytic and injective on \( U \setminus E(q) \) where

\[
E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\}
\]

and are such that \( q'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(q) \). Further let the subclass of \( F \) for which \( q(0) = a \) be denoted by \( F(a) \), \( F(0) = F_0 \) and \( F(1) = F_1 \).

**Definition 2** [15]. A function \( L(z,t) \) (\( z \in U, t \geq 0 \)) is said to be a subordination chain if \( L(.,t) \) is analytic and univalent in \( U \) for all \( t \geq 0 \), \( L(z,.) \) is continuously differentiable on \([0,1]\) for all \( z \in U \) and \( L(z,t_1) \prec L(z,t_2) \) for all \( 0 \leq t_1 \leq t_2 \).

**Lemma 1** [16]. The function \( L(z,t) : U \times [0;1] \to \mathbb{C} \) of the form

\[
L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots \quad (a_1(t) \neq 0; t \geq 0)
\]

and \( \lim_{t \to \infty} |a_1(t)| = \infty \) is a subordination chain if and only if

\[
\text{Re} \left\{ \frac{z \partial L(z,t) / \partial z}{\partial L(z,t) / \partial t} \right\} > 0 \quad (z \in U, t \geq 0).
\]

**Lemma 2** [12]. Suppose that the function \( \mathcal{H} : \mathbb{C}^2 \to \mathbb{C} \) satisfies the condition

\[
\text{Re} \left\{ \mathcal{H}(is;t) \right\} \leq 0
\]

for all real \( s \) and for all \( t \leq -n \left( 1 + s^2 \right) / 2, n \in \mathbb{N} \). If the function \( p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots \) is analytic in \( U \) and

\[
\text{Re} \left\{ \mathcal{H} \left( p(z); zp'(z) \right) \right\} > 0 \quad (z \in U),
\]
Lemma 3 [13]. Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in H(U)$ with $h(0) = c$. If $\Re \{\kappa b(z) + \gamma\} > 0 (z \in U)$, then the solution of the following differential equation:

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; q(0) = c)$$

is analytic in $U$ and satisfies $\Re \{\kappa q(z) + \gamma\} > 0$ for $z \in U$.

Lemma 4 [14]. Let $p \in F(a)$ and let $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ be analytic in $U$ with $q(z) \neq a$ and $n \geq 1$. If $q$ is not subordinate to $p$, then there exists two points $z_0 = r_0 e^{i\theta} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$ such that

$$q(U_{r_0}) \subset p(U); \quad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 p'(z_0) = m \zeta_0 p'(\zeta_0) \quad (m \geq n).$$

Lemma 5 [15]. Let $q \in H[a; 1]$ and $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}$. Also set $\varphi \left( q(z), zq'(z) \right) = h(z)$. If $L(z, t) = \varphi \left( q(z), tzq'(z) \right)$ is a subordination chain and $p \in H[a; 1] \cap F(a)$, then

$$h(z) \prec \varphi \left( p(z), zp'; (z) \right),$$

implies that $q(z) \prec p(z)$. Furthermore, if $\varphi \left( q(z), zq'(z) \right) = h(z)$ has a univalent solution $q \in F(a)$, then $q$ is the best subordinant.

In the present paper, we aim to prove some subordination-preserving and superordination-preserving properties associated with the integral operator $J_{p,s,b}$. Sandwich-type result involving this operator is also derived.

2. Main results

Unless otherwise mentioned, we assume throughout this section that $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$, $\Re (b) > 0$, $p \in \mathbb{N}$ and $z \in U$.

Theorem 1. Let $f, g \in A(p)$ and

$$\Re \left\{ 1 + \frac{z\phi''(z)}{\phi(z)} \right\} > -\delta \quad \left( \phi(z) = \frac{J_{p,s-1,b}(g)(z)}{z^p}; z \in U \right),$$

where $\delta$ is given by

$$\delta = \frac{1 + |b + 1|^2 - \left| 1 - (b + 1)^2 \right|}{4 \left[ 1 + \Re (b) \right]} \quad (z \in U).$$

Then the subordination condition

$$\frac{J_{p,s-1,b}(f)(z)}{z^p} = \frac{J_{p,s-1,b}(g)(z)}{z^p}$$

(2.3)
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implies that
\[ \frac{J_{p,s,b}(f)(z)}{z^p} < \frac{J_{p,s,b}(g)(z)}{z^p} \]  
(2.4)

and the function \( \frac{J_{p,s,b}(g)(z)}{z^p} \) is the best dominant.

**Proof.** Let us define the functions \( F(z) \) and \( G(z) \) in \( U \) by
\[ F(z) = \frac{J_{p,s,b}(f)(z)}{z^p} \quad \text{and} \quad G(z) = \frac{J_{p,s,b}(g)(z)}{z^p} \quad (z \in U) \]  
(2.5)

and without loss of generality we assume that \( G(z) \) is analytic, univalent on \( \bar{U} \) and
\[ G'(\zeta) \neq 0 \quad (|\zeta| = 1). \]

If not, then we replace \( F(z) \) and \( G(z) \) by \( F(\rho z) \) and \( G(\rho z) \), respectively, with \( 0 < \rho < 1 \). These new functions have the desired properties on \( \bar{U} \), so we can use them in the proof of our result and the results would follow by letting \( \rho \to 1 \).

We first show that, if
\[ q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U), \]  
(2.6)

then
\[ \Re\{q(z)\} > 0 \quad (z \in U). \]

From (1.10) and the definition of the functions \( G, \phi \), we obtain that
\[ \phi(z) = G(z) + \frac{zG'(z)}{b+1}. \]  
(2.7)

Differentiating both sides of (2.7) with respect to \( z \) yields
\[ \phi'(z) = \left(1 + \frac{1}{b+1}\right)G'(z) + \frac{zG''(z)}{b+1}. \]  
(2.8)

Combining (2.6) and (2.8), we easily get
\[ 1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + b + 1} = h(z) \quad (z \in U). \]  
(2.9)

It follows from (2.1) and (2.9) that
\[ \Re\{h(z) + b + 1\} > 0 \quad (z \in U). \]  
(2.10)

Moreover, by using Lemma 3, we conclude that the differential equation (2.9) has a solution \( q(z) \in H(\bar{U}) \) with \( h(0) = q(0) = 1 \). Let
\[ \mathcal{H}(u, v) = u + \frac{v}{u + b + 1} + \delta, \]
where $\delta$ is given by (2.2). From (2.9) and (2.10), we obtain $\text{Re} \left\{ \mathcal{H} \left( q(z); z q'(z) \right) \right\} > 0 \ (z \in U)$.

To verify the condition

$$\text{Re} \left\{ \mathcal{H} (i \vartheta; t) \right\} \leq 0 \quad \left( \vartheta \in \mathbb{R}; \ t \leq -\frac{1 + \vartheta^2}{2} \right),$$

we proceed as follows:

$$\text{Re} \left\{ \mathcal{H} (i \vartheta; t) \right\} = \text{Re} \left\{ i \vartheta + \frac{t}{b + 1 + i \vartheta} + \delta \right\} = \frac{t (1 + \text{Re} (b))}{|b + 1 + i \vartheta|^2} + \delta \leq -\frac{\Upsilon (b, \vartheta, \delta)}{2 |b + 1 + i \vartheta|^2},$$

where

$$\Upsilon (b, \vartheta, \delta) = [1 + \text{Re} (b) - 2 \delta] \vartheta^2 - 4 \delta \text{Im} (b) \vartheta - 2 \delta |b + 1| + 1 + \text{Re} (b).$$

For $\delta$ given by (2.2), the coefficient of $\vartheta^2$ in the quadratic expression $\Upsilon (b, \vartheta, \delta)$ given by (2.12) is positive or equal to zero. To check this, put $b + 1 = c$, so that

$$1 + \text{Re} (b) = c_1 \quad \text{and} \quad \text{Im} (b) = c_2.$$

We thus have to verify that

$$c_1 - 2 \delta \geq 0,$$

or

$$c_1 \geq 2 \delta = \frac{1 + |c|^2 - |1 - c^2|}{2c_1}.$$

This inequality will hold true if

$$2c_1^2 + |1 - c^2| \geq 1 + |c|^2 = 1 + c_1^2 + c_2^2,$$

that is, if

$$|1 - c^2| \geq 1 - \text{Re} (c^2),$$

which is obviously true. Moreover, the quadratic expression $\Upsilon (b, \vartheta, \delta)$ by $\vartheta$ in (2.12) is a perfect square for the assumed value of $\delta$ given by (2.2). Hence we see that (2.11) holds.

Thus, by using Lemma 2, we conclude that

$$\text{Re} \left\{ q(z) \right\} > 0 \quad (z \in U),$$

that is, that $G$ defined by (2.5) is convex (univalent) in $U$. Next, we prove that the subordination condition (2.3) implies that

$$F(z) \prec G(z),$$
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for the functions $F$ and $G$ defined by (2.5). Consider the function $L(z, t)$ given by

$$L(z, t) = G(z) + \frac{(1 + t) z G'(z)}{b + 1} \quad (0 \leq t < \infty; z \in U).$$

(2.13)

We note that

$$\frac{\partial L(z, t)}{\partial z} \bigg|_{z=0} = G'(0) \left(1 + \frac{1 + t}{b + 1}\right) \neq 0 \quad (0 \leq t < \infty; z \in U; \text{Re}\{b + 1\} > 0).$$

This shows that the function $L(z, t) = a_1(t) z + \ldots$, satisfies the condition $a_1(t) \neq 0 \quad (0 \leq t < \infty)$. Further, we have

$$\text{Re} \left\{ \frac{z \partial L(z, t)}{\partial z} / \partial t \right\} = \text{Re} \{b + 1 + (1 + t) q(z)\} > 0 \quad (0 \leq t < \infty; z \in U).$$

Since $G(z)$ is convex and $\text{Re}\{b + 1\} > 0$. Therefore, by using Lemma 1, we deduce that $L(z, t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{z G'(z)}{b + 1} = L(z, 0)$$

and

$$L(z, 0) \prec L(z, t) \quad (0 \leq t < \infty),$$

which implies that

$$L(\zeta, t) \notin L(U, 0) = \phi(U) \quad (0 \leq t < \infty; \zeta \in \partial U).$$

(2.14)

If $F$ is not subordinate to $G$, by using Lemma 4, we know that there exist two points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1 + t) \zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty).$$

(2.15)

Hence, by using (2.5), (2.13), (2.14) and (2.3), we have

$$L(\zeta_0, t) = G(\zeta_0) + \frac{(1 + t) \zeta_0 G'(\zeta_0)}{b + 1} = F(z_0) + \frac{z_0 F'(z_0)}{b + 1} = \frac{J_{p, s-1, b}(f)(z_0)}{z_0^p} \in \phi(U).$$

This contradicts (2.14). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that the function $G$ is the best dominant. This completes the proof of Theorem 1.

We now derive the following superordination result.

**Theorem 2.** Let $f, g \in A(p)$ and

$$\text{Re} \left\{ 1 + \frac{z \phi''(z)}{\phi'(z)} \right\} > -\delta \quad \left( \phi(z) = \frac{J_{p, s-1, b}(g)(z)}{z^p}; z \in U \right),$$

(2.16)
where $\delta$ is given by (2.2). If the function $\frac{J_{p,s-1,b}(f)(z)}{z^p}$ is univalent in $U$ and $\frac{J_{p,s,b}(f)(z)}{z^p} \in F$, then the superordination condition

$$\frac{J_{p,s-1,b}(g)(z)}{z^p} \prec \frac{J_{p,s-1,b}(f)(z)}{z^p}$$

implies that

$$\frac{J_{p,s,b}(g)(z)}{z^p} \prec \frac{J_{p,s,b}(f)(z)}{z^p}$$

and the function $\frac{J_{p,s,a}(g)(z)}{z^p}$ is the best subordinant.

**Proof.** Suppose that the functions $F, G$ and $q$ are defined by (2.5) and (2.6), respectively. By applying similar method as in the proof of Theorem 1, we get

$$\Re \{ q(z) \} > 0 \quad (z \in U).$$

Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L(z, t)$ is defined by (2.13). Since $G$ is convex, by applying a similar method as in Theorem 1, we deduce that $L(z, t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{zG'(z)}{b + 1} = \varphi \left( G(z), zG'(z) \right)$$

has a univalent solution $G$, it is the best subordinant. This completes the proof of Theorem 2.

Combining the above-mentioned subordination and superordination results involving the operator $J_{p,s,b}$, the following "sandwich-type result" is derived.

**Theorem 3.** Let $f, g_j \in A(p)$ ($j = 1, 2$) and

$$\Re \left\{ 1 + \frac{z\phi_j''(z)}{\phi_j'(z)} \right\} > -\delta \quad \left( \phi_j(z) = \frac{J_{p,s-1,b}(g_j)(z)}{z^p} \right) (j = 1, 2); \; z \in U,$$

where $\delta$ is given by (2.2). If the function $\frac{J_{p,s-1,b}(f)(z)}{z^p}$ is univalent in $U$ and $\frac{J_{p,s,b}(f)(z)}{z^p} \in F$, then the condition

$$\frac{J_{p,s-1,b}(g_1)(z)}{z^p} \prec \frac{J_{p,s-1,b}(f)(z)}{z^p} \prec \frac{J_{p,s-1,b}(g_2)(z)}{z^p}$$

implies that

$$\frac{J_{p,s,b}(g_1)(z)}{z^p} \prec \frac{J_{p,s,b}(f)(z)}{z^p} \prec \frac{J_{p,s,b}(g_2)(z)}{z^p}$$

and the functions $\frac{J_{p,s,b}(g_1)(z)}{z^p}$ and $\frac{J_{p,s,b}(g_2)(z)}{z^p}$ are, respectively, the best subordinant and the best dominant.

**Remark.** (i) Putting $b = p$ and $s = \alpha$ ($\alpha \geq 0, p \in \mathbb{N}$) in our results of this paper, we obtain the results obtained by Aouf and Seoudy [2];

(ii) Specializing the parameters $s$ and $b$ in our results of this paper, we obtain the results for the corresponding operators $F_{\nu,p}$, $J^m_p$ and $J^m_p(l)$ which are defined in the introduction.
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