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DUAL PROBABILISTIC ANALYSIS OF THE TRANSIENT HEAT TRANSFER BY THE STOCHASTIC FINITE ELEMENT METHOD WITH OPTIMIZED POLYNOMIAL BASIS

The main aim of this work is to contrast three various probabilistic computational techniques, namely analytical, simulation and perturbation-based, in a solution of the transient heat transfer problem in specific axisymmetric problem with Gaussian uncertainty in physical parameters. It is done thanks to a common application of the Finite Element Method program *ABAQUS* (for the deterministic part) and symbolic algebra system *MAPLE*, where all probabilistic procedures have been programmed. We determine up to the fourth order probabilistic characteristics of the resulting temperatures, i.e. expectations, coefficients of variation, skewness and kurtosis together with the histograms – all as the functions of the input coefficient of variation of random heat conductivity coefficient. Stochastic perturbation technique is implemented here using the tenth order Taylor series expansion and traditional Least Squares Method released with polynomial basis whose final order is a subject of the separate statistical optimization. Probabilistic results computed show almost perfect agreement of all the probabilistic characteristics under consideration, which means that the traditional simulation method may be replaced due to the time and computer scale savings with the stochastic perturbation method.

Keywords: heat transfer; Stochastic Finite Element Method; Monte-Carlo simulation; stochastic perturbation technique

1. Introduction

Heat transfer phenomena in both statistically homogeneous and heterogeneous media include a number of uncertainty sources as stochastic fluctuations and waviness of boundaries and boundary conditions, natural unpredictability of physical properties of solids, fluids, gases or their mixtures as well as their material parameters. Such phenomena were under consideration since many years, cf.

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Chorin (1974) [1] and Emery (2004) [2], also in the context of various couplings with fluid flow or mass transfer. There exists a number of different techniques available to determine probabilistic moments of the temperature distribution (and its time fluctuation) as the Monte-Carlo simulation (Binder and Heermann (1977) [3]), polynomial chaos and Karhunen-Loeve expansions (Xiu and Karniadakis (2003) [4]), stochastic perturbation methods, see Kamiński (2013) [5], and even interval analysis approach engaged by Wang and Qiu (2015) [6]. It is widely known that traditional Monte-Carlo simulation is usually the very large time consuming probabilistic computational method, especially in the view of highly unstable heating or freezing processes that demand a huge number of time steps until steady state is reached.

That is why a reliable numerical stochastic method of significantly lower overall cost is required, which additionally allows for a determination of higher order statistics to enable for a recognition of probability distribution of the resulting temperature or heat flux. Therefore, the main aim of this paper is to demonstrate an implementation of the transient heat transfer analysis with Gaussian random parameters by using of the generalized stochastic perturbation technique. Time and computer power consumption in this method is relatively low with respect to any other existing probabilistic method. Its implementation is based on the Response Function Method, where nodal temperatures and their time fluctuations are approximated by random polynomials whose orders and coefficients are determined and optimized via separate FEM transient heat transfer experiments. Duality of this approach originates from the fact that it is possible to integrate these polynomials according to classical definitions of the probability theory and, alternatively, to include them into the Taylor series expansion to calculate basic probabilistic moments of the desired state parameters. This approach is illustrated with the use of statistically homogenous circular conductor having random conductivity and it is modeled by the three alternative probabilistic techniques - semi-analytical, simulation-based and the one adjacent to the generalized stochastic perturbation technique.

Deterministic transient heat transfer problem is solved numerically by using of the Finite Element Method program *ABAQUS/CEA* ver. 6.10 and also analytically thanks to the classical series solution. All the probabilistic procedures adjacent to a combination of two deterministic and three entirely different probabilistic method are implemented in the computer algebra system *MAPLE*, v. 17. We assume that heat conductivity coefficient k has Gaussian distribution with specified expectation and some a priori given variability interval for its coefficient of variation, $\alpha(k) \in [0.00, 0.15]$. We contrast here the basic probabilistic characteristics of the resulting temperature, i.e. its expectations, coefficients of variations, skewness and kurtosis at the steady-state conditions for in a specific point of the heated disk. An observed very good coincidence in-between these methods and their results makes the proposed dual probabilistic computational technique very

attractive and promising in further computational physics experiments. One needs to mention that this methodology may find its application to stochastic fire simulation similar to the considerations provided by Sakji, Soize and Heck (2009) [7].

2. Probabilistic analysis of the heat transfer problem

Generally, transient heat flow problem consists in determining the temperature field $\theta = \theta(x, \tau)$ governed by the following differential equation:

$$\rho c \dot{\theta} - \left(k_{ij}\theta_{,j}\right)_{,i} - g = 0; x_i \in \Omega; \tau \in [0, \infty), i=1,2,3, \tag{1}$$

where c is the heat capacity characterizing the region Ω, ρ is the density of the material contained in Ω, k_{ij} is thermal conductivity tensor, while g is the rate of heat generated per unit volume; variables θ and τ denote temperature field values and time, respectively. This equation should fulfil the boundary conditions on the additional subsets of the external boundary $\partial\Omega$, that are given as follows:

1) temperature (essential) boundary conditions

$$\theta = \tilde{\theta}; x_i \in \partial\Omega_\theta, \tag{2}$$

and for $\partial\Omega_q$ part of the total $\partial\Omega$:

2) heat flux (natural) boundary conditions

$$\frac{\partial\theta}{\partial n} = \hat{q}; \; x_i \in \partial\Omega_q, \tag{3}$$

where $\partial\Omega_\theta \cup \partial\Omega_q = \partial\Omega$ and $\partial\Omega_\theta \cap \partial\Omega_q = \{\varnothing\}$. Initial conditions are proposed here as

$$\theta^0 = \theta(x_i; 0); \; x_i \in \Omega. \tag{4}$$

Let us consider further some continuous temperature variations $\delta\theta(x_i)$ defined in the interior of the region Ω and vanishing on $\partial\Omega_\theta$. Multiplying Eqn. (4) by the test function specified and integrating it over the entire Ω, we obtain

$$\int_{\Omega}\left(\rho c\,\dot{\theta}-\left(k_{ij}\theta_{,j}\right)_{,i}-g\right)\delta\theta\;d\Omega=0\,;\;\; x_i\in\Omega;\quad \tau\in[0,\infty)\,. \tag{5}$$

It is well known that this formulation is frequently rewritten as

$$\begin{aligned}&\int_{\Omega}\left(\rho c\,\dot{\theta}\delta T+k_{ij}\theta_{,j}\delta\theta_{,i}-g\delta\theta\right)d\Omega-\int_{\partial\Omega_q}\hat{q}\delta\theta d(\partial\Omega)=0\,;\\ &x_i\in\Omega;\;\tau\in[0,\infty)\,.\end{aligned} \tag{6}$$

The equation stated below is the transient formulation of the principle of virtual temperatures and is used to provide its stochastic perturbation technique counterpart relevant to the generalized stochastic perturbation technique [4]. We use for this purpose Taylor series expansion of random temperatures process as

$$\theta(k,\tau)=\theta^0\left(k^0,\tau\right)+\sum_{m=1}^{n}\frac{\varepsilon^m}{m!}\frac{\partial^m\theta(k,\tau)}{\partial k^m}\bigg|_{k=k^0}\left(k-k^0\right)^m, \tag{7}$$

where k is heat conductivity coefficient characterizing isotropic media, n stands for the order of stochastic expansion, while superscript $(.)^0$ means the mean values of the given parameter or state function. This Taylor series representation is inserted into the basic definitions, i.e. expected values $E[\theta(k,\tau)]$, standard deviations $\sigma(\theta(k,\tau))$, coefficient of variation $\alpha(\theta(k,\tau))$, skewness $\beta(\theta(k,\tau))$ and kurtosis $\kappa(\theta(k,\tau))$ to develop analytical perturbation-based formulas at the given time τ as

$$\begin{aligned}&E[\theta(k,\tau)]=\int_{-\infty}^{+\infty}\theta(k,\tau)\,p_k(x)dx,\\ &\sigma(\theta(k,\tau))=\left\{\int_{-\infty}^{+\infty}(\theta(k,\tau)-E[\theta(k,\tau)])^2\,p_k(x)dx\right\}^{\frac{1}{2}},\end{aligned} \tag{8}$$

$$\begin{aligned}&\alpha(\theta(k,\tau))=\frac{\sigma(\theta(k,\tau))}{E[\theta(k,\tau)]},\;\beta(\theta(k,\tau))=\frac{\mu_3(\theta(k,\tau))}{\sigma^3(\theta(k,\tau))},\\ &\kappa(\theta(k,\tau))=\frac{\mu_4(\theta(k,\tau))}{\sigma^4(\theta(k,\tau))}-3.\end{aligned} \tag{9}$$

Obviously, $\mu_3(\theta(k,\tau))$, $\mu_4(\theta(k,\tau))$ denote here the third and the fourth central probabilistic moments of the temperature, while $p_k(x)$ is the probability density function (PDF) of random heat conductivity (postponed in full analytical version for a brevity of the presentation). Practical engineering computations include ε=1 and demand determination of higher order partial derivatives of the resulting temperature with respect to the randomized heat capacity. These are calculated from the hierarchical equations - the set of the algebraic equations of the systematically increasing order (from 0th up to the *n*th). Recursive form of these equations is demonstrated below with

- zeroth-order partial differential equation

$$\int_{\Omega}\left(\rho^0 c^0\,\dot{\theta}^0\,\delta T + k_{ij}^0\theta_{,j}^0\delta\theta_{,i}\right)d\Omega = \int_{\partial\Omega_q}\hat{q}^0\,\delta\theta\,d(\partial\Omega) + \int_{\Omega} g^0\,\delta\theta\,d\Omega\,, \tag{10}$$

- as well as the *n*th order equation

$$\begin{aligned}&\int_{\Omega}\left(\sum_{l=0}^{n}\binom{n}{k}\left(\sum_{m=0}^{l}\binom{l}{m}\frac{\partial^l\rho}{\partial k^l}\frac{\partial^{l-m}c}{\partial k^{l-m}}\right)\frac{\partial^{n-l}\dot{\theta}}{\partial k^{n-l}}\right)\delta\theta\,d\Omega\\ &+\int_{\Omega}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{\partial^l k_{ij}}{\partial k^l}\frac{\partial^{n-l}\theta_{,j}}{\partial k^{n-l}}\right)\delta\theta_{,i}\,d\Omega = \int_{\partial\Omega_q}\frac{\partial^n\hat{q}}{\partial k^n}\delta\theta\,d(\partial\Omega) + \int_{\Omega}\frac{\partial^n g}{\partial k^n}\,\delta\theta\,d\Omega\end{aligned} \tag{11}$$

Further simplifications in this particular case study for the last equation are remarkable as most of the partial derivatives with respect to the input random variable simply vanish. Finally, having solved these equations for θ^0 and for up to the *n*th order partial derivatives, respectively, we derive the expressions for the expected values and higher probabilistic moments and the coefficients for temperature field and its time fluctuations.

3. Stochastic Finite Element Method equations

Let us assume that the region Ω is discretized by the use of the set of finite elements and that the scalar temperature field $\theta(x_i)$ is described by the nodal temperatures vector T_α

$$\theta(x_i) = \varphi_\alpha(x_i)\,T_\alpha;\ i=1,2,3;\ \alpha=1,2,\ldots,N, \tag{12}$$

where N is the total number of degrees of freedom introduced. The temperature derivatives can be written in the form

$$\theta_{,i} = \varphi_{\alpha,i} T_\alpha \,,\ i=1,2,3. \tag{13}$$

Moreover, let us introduce the capacity matrix $C_{\alpha\beta}$, the heat conductivity matrix $K_{\alpha\beta}$ and the vector P_α as follows:

$$\begin{aligned} &C_{\alpha\beta} = \int_\Omega \rho c \varphi_\alpha \varphi_\beta \, d\Omega \,,\ K_{\alpha\beta} = \int_\Omega k_{ij} \varphi_{\alpha,i} \varphi_{\beta,j} \, d\Omega \,, \\ &P_\alpha = \int_\Omega g \varphi_\alpha \, d\Omega + \int_{\partial\Omega} \hat{q} \varphi_\alpha \, d\Omega \,. \end{aligned} \tag{14}$$

Next, let us introduce these matrixes into the additional variational formulation we obtain the following algebraic equations system:

$$C_{\alpha\beta} \dot{T}_\beta + K_{\alpha\beta} T_\beta = P_\alpha \,. \tag{15}$$

The main issue in transient problems is the additional time discretization using some time increment Δt. Then we can rewrite the last equation in the following manner:

$$C_{\alpha\beta} \frac{T_\beta(t+\Delta t) - T_\beta(t)}{\Delta t} + K_{\alpha\beta} T_\beta(t) = P_\alpha \,. \tag{16}$$

Considering the second component in this statement we obtain the explicit method, where the nodal temperatures vector in this component is taken at the beginning of this time step. However, it is possible to introduce the extra coefficient $0 \le \delta \le 1$ to include in this term the temperatures vector after the time step also. There holds

$$C_{\alpha\beta} \frac{T_\beta(t+\Delta t) - T_\beta(t)}{\Delta t} + K_{\alpha\beta} \left\{ \delta T_\beta(t+\Delta t) + (1-\delta) T_\beta(t) \right\} = P_\alpha \,, \tag{17}$$

where $\delta = 0$ is equivalent to the explicit method, $\delta = 1/2$ serves for the Crank-Nicholson method, $\delta=2/3$ stands for the Galerkin method and at last $\delta = 1$ is used in the implicit method (one can use this algorithm with δ as the extra input parameter); there are also three level schemes, where the temperatures in the moments $t+\Delta t, t, t-\Delta t$ are included at once.

Analogously to the previous considerations (provided also for the second order analysis before) we obtain the following systems of algebraic equations describing the generalized stochastic formulation of the transient heat flow problem:

- zeroth-order equation

$$C^0_{\alpha\beta}\dot{T}^0_\beta + K^0_{\alpha\beta}T^0_\beta = P^0_\alpha, \tag{18}$$

and the nth order equation

$$C^0_{\alpha\beta}\frac{\partial^n \dot{T}_\beta}{\partial b^n} + K^0_{\alpha\beta}\frac{\partial^n T_\beta}{\partial b^n} = \frac{\partial^n P_\alpha}{\partial b^n} - \sum_{m=1}^{n}\binom{n}{m}\frac{\partial^m C_{\alpha\beta}}{\partial b^m}\frac{\partial^{n-m}\dot{T}_\beta}{\partial b^{n-m}} - \sum_{m=1}^{n}\binom{n}{m}\frac{\partial^m K_{\alpha\beta}}{\partial b^m}\frac{\partial^{n-m} T_\beta}{\partial b^{n-m}}, \tag{19}$$

As it is clear now, the DDM version needs a formation and the solution of the increasing order equations obtained from the initial one by a systematic differentiation with respect to the random input variable provided in a quite deterministic way. It is also clear that the probabilistic transient problem needs successive polynomial responses from time increment to the time increment, therefore for a discrete time moment τ we introduce the following polynomial approximation:

$$T_\beta(\tau) = D^\tau_{\beta m}(\tau) b^m,\ m=0,\ldots,n\text{-}1;\ \beta=1,\ldots,N,\ \tau\in[0,\infty). \tag{20}$$

Hence, it yields

$$\theta(x_i,\tau) = \varphi_\beta(x_i)T_\beta(\tau) = \varphi_\beta(x_i)\,D^\tau_{\beta m}(\tau)b^m;\quad i=1,2,3;\ \alpha=1,2,...,N,\ m=0,\ldots,n\text{-}1;\ \tau\in[0,\infty). \tag{21}$$

Therefore, the temperature gradients are similarly determined as

$$\theta_{,j}(\tau) = \varphi_{\beta,j}T_\beta(\tau) = \varphi_{\beta,j}D^\tau_{\beta m}(\tau)b^m,\ i=1,2,3,\ m=0,\ldots,n\text{-}1;\ \tau\in[0,\infty). \tag{22}$$

Optimization of this local polynomial basis order *n* is inherent in its Least Squares Method approximation. An optimum choice is equivalent to the polynomial demonstrating extreme correlation to the set of FEM experiments (preferably should equal to 1). This polynomial should minimize at the same time the fitting variance and the RMS error of the entire fitting procedure (an illustration of this process in included in Table 1 below). Finally, one realizes that the temperature-dependent physical parameters may lead to further numerical complications in the SFEM implementation for transient problems using the Response Function Method even with polynomial basis.

4. Numerical analysis

We consider statistically homogeneous heat convector of the following circular solid domain (Fig. 1) where non-stationary heat transfer given by the Fourier law based on polar coordinate system (r,φ) is analyzed

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}=K\frac{\partial\theta}{\partial t}, \tag{23}$$

where $K=c\rho/k$ includes in turn heat capacity *c*, mass density ρ as well as heat conductivity coefficient *k*. The boundary conditions are set on the outer surfaces of this axisymmetric region ($T(r=R_1)=T_1$ and $T(r=R_2)=T_2$) and are all fully deterministic.

As it is known, cf. Carlsaw and Jaeger (1959) [8], this problem has some analytical solution expressed by the Bessel functions that can be rewritten as

$$\begin{aligned}\theta(r,t)=\theta_1+(\theta_2-\theta_1)\cdot\frac{\ln(r)-\ln(R_1)}{\ln(R_2)-\ln(R_1)}\\ -\pi\sum_{n=1}^{\infty}\frac{J_0(a_nR_1)\cdot Y_0(a_nr)-J_0(a_nr)\cdot Y_0(a_nR_1)}{J_0(a_nR_1)^2-J_0(a_nR_2)^2}\\ \times\cdot J_0(a_nR_2)\{\theta_1J_0(a_nR_2)-\theta_2J_0(a_nR_1)\}e^{-Ka_nt}\end{aligned} \tag{24}$$

where $T_i=T\Big|_{r=R_i}, i=1,2$, and the coefficients a_n are non-negative square roots for the following algebraic equation:

$$J_0(aR_1)\cdot Y_0(aR_2)-J_0(aR_2)\cdot Y_0(aR_1)=0 \tag{25}$$

while J_0, Y_0 are the well-known Bessel functions of the first kind.

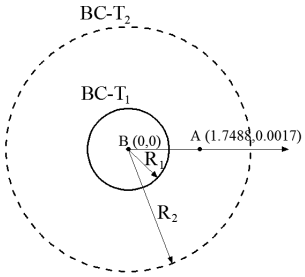

Fig. 1. Computational domain with the boundary conditions

Rys. 1. Geometria modelowanej struktury wraz z warunkami brzegowymi

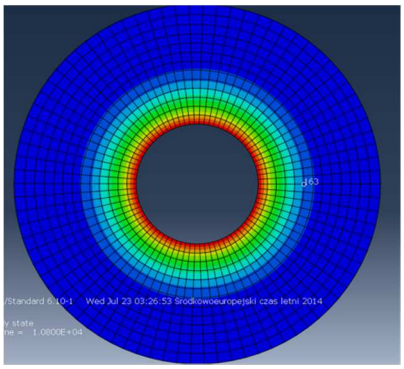

Fig. 2. Spatial discretization of the problem in the system ABAQUS

Rys. 2. Dyskretyzacja w programie MES ABAQUS

Analytical solution contrasted further in computational experiments is obtained in the program *MAPLE* with the first 318 expansion terms here and for the boundary temperatures equal to $T_1 = 100\left[^\circ C\right]$, $T_2 = 0\left[^\circ C\right]$, correspondingly. The following material properties are adopted (adjacent to the stainless steel plate in room temperature): $c = 440\left[\frac{J}{m^4 K}\right]$, $\rho = 7800\left[\frac{kg}{m^3}\right]$, while k is the input Gaussian random variable with the expectation $E[k] = 58\left[\frac{W}{m^2 K}\right]$ and coefficient of variation $\alpha(k) \in [0.00, 0.15]$. A series of the computational FEM solutions has been obtained with the system ABAQUS CEA, ver. 6.10 and its 4-noded linear heat transfer quadrilaterals (DC2D4 type) - spatial discretization includes 13 strips in radial direction and 82 finite elements on the inner and outer circumference (1088 in total) (Fig. 2). Time stepping procedure is based on the fundamental increment Δt=108 secs and includes 100 increments until the steady-state conditions. The resulting temperature, whose probabilistic moments are contrasted further, has been obtained at the radius equal to r=1.74887 [m]. The Least Squares Method has been programmed with the use of Taylor-Newton-Gauss algorithm that enables for direct computation of the correlation of the series of trial points with the fitting polynomial function as well as RMS error of this procedure, total sum of the residuals squares and the curve fitting variance. Such a statistical optimization has been carried out for the polynomials of the degrees from the ninth to the second one and it is based upon maximization of the correlation and minimization of the RMS error and the fitting variance at the same time; the additional data justifying the choice of the fifth order are collected in Table 1. A comparative Monte-Carlo simulation has been performed with the use of 10^5 random trials and it enables to contrast input with the output histograms of heat conductivity coefficient and the resulting temperatures for $\alpha(k)$=0.06 (Fig. 3) and $\alpha(k)$=0.15 (Fig. 4). The left histograms include both linear trends adjacent to the theoretical PDF shape; one may easily discover high

Table 1. Statistical optimization of the random polynomial basis order

Tabela 1. Optymalizacja statystyczna losowego wielomianu

Polynomial order	Correlation	RMS error	Squares sum	Variance
9	-0,430867	1,10017E11	1,33141E23	1,33404E22
8	0,714752	6,08909E8	4,07847E18	2,29869E19
7	0,276704	2,58471E7	7,34881E15	1,49208E15
6	0,999512	0,152492	0,255802	0,248877E-1
5	1,000000	0,312325E-2	0,107128E-3	0,100123E-4
4	0,999989	0,227047E-1	0,566403E-2	0,567674E-3
3	0,999678	0,122030	0,163791	0,163738E-1
2	0,995632	0,462267	2,35068	0,233968

quality of the Gaussian variable internal generator in the system *MAPLE*. It is remarkable that all the density functions are very close in their shapes to the Gaussian distributions. One may observe that the final PDF of the steady state temperature for $\alpha(k)=0.15$ (right graph in Fig. 4) shows some apparent non-zero skewness contrary to the PDF determined with a smaller value of the coefficient $\alpha(k)$.

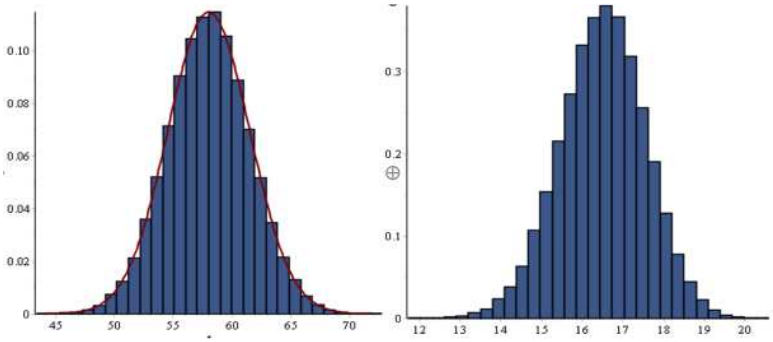

Fig. 3. Histograms of input and output random variables for $\alpha = 0.06$

Rys. 3. Histogramy wejściowej i wynikowej zmiennej losowej dla $\alpha = 0.06$

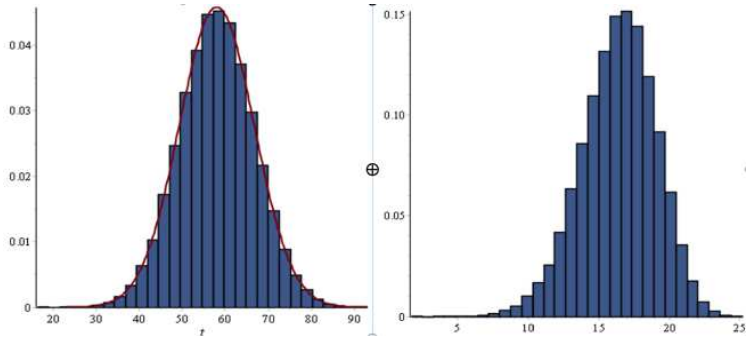

Fig. 4. Histograms of input and output random variables $\alpha = 0.15$

Rys. 4. Histogramy wejściowej i wynikowej zmiennej losowej dla $\alpha = 0.15$

Further, we compute in turn the expected values, coefficients of variation, skewness and kurtosis for the polynomial of the fifth degree and depending on α in the range of 0,0÷0,15 and these statistical parameters are collected in Figs. 5-8. The results are summarized for both Stochastic Finite Element Method (SFEM)

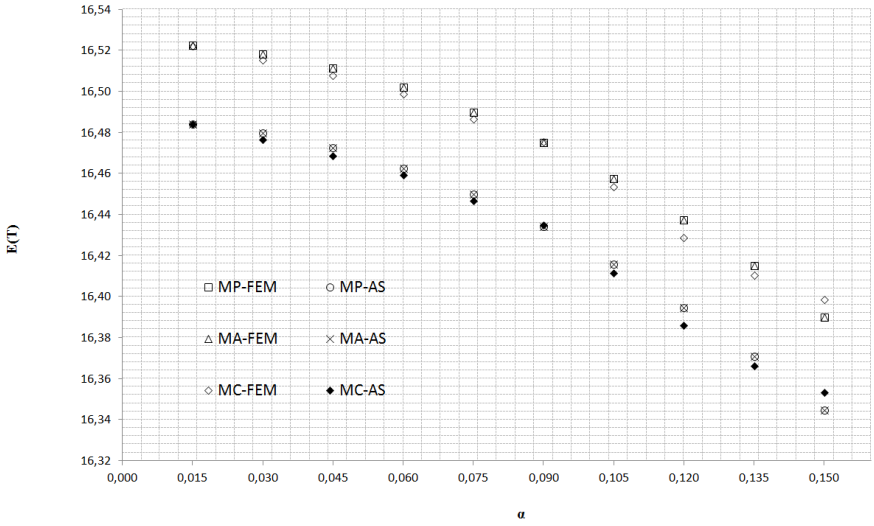

Fig. 5. Expected values of the temperature T_A as a function of input coefficient of variation α

Rys. 5. Wartości oczekiwane temperatury T_A w funkcji wejściowego współczynnika wariancji α

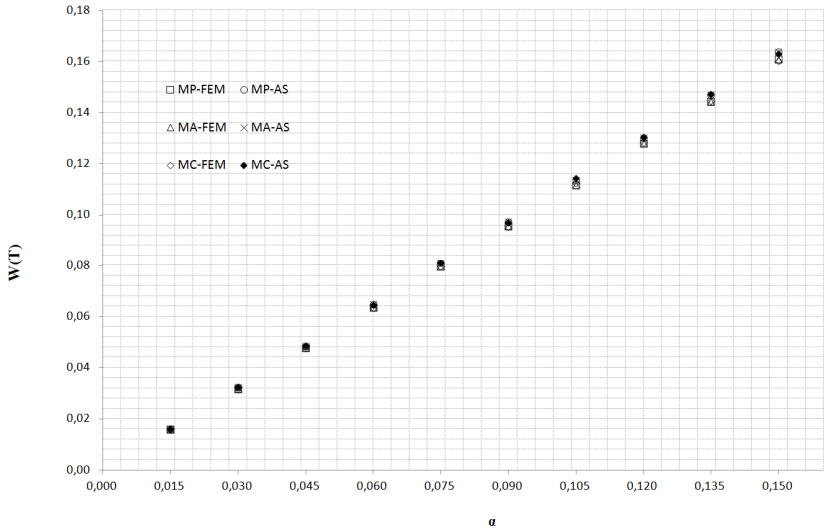

Fig. 6. Coefficients of variation of the temperature T_A as a function of input coefficient of variation α

Rys. 6. Współczynniki wariancji temperatury T_A w funkcji wejściowego współczynnika wariancji α

and analytical solution (AS) in such a way that additional abbreviation *MP* is adequate to the results obtained by the tenth order stochastic perturbation method, *MA* - represents the results obtained by analytical integration directly from the probability theory definitions and, consecutively, *MC* stands here for the results estimated via the Monte Carlo simulation scheme.

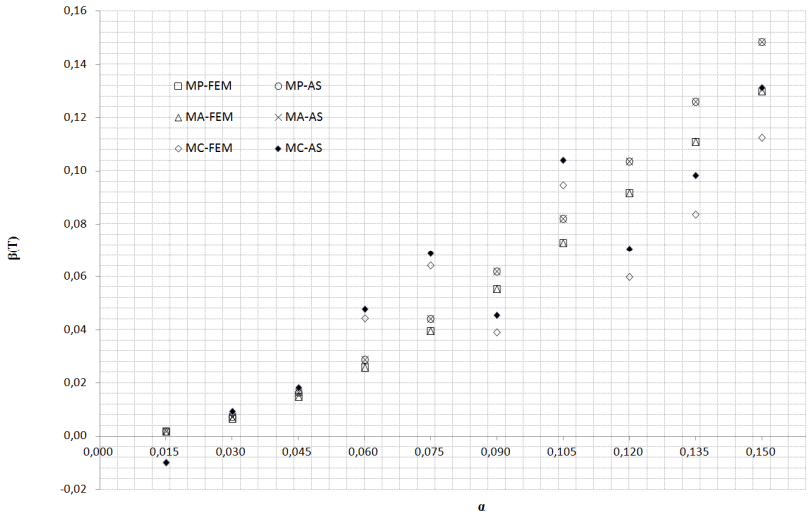

Fig. 7. Skewness of the temperature T_A as a function of input coefficient of variation α

Rys. 7. Skośność temperatury TA w funkcji wejściowego współczynnika wariancji α

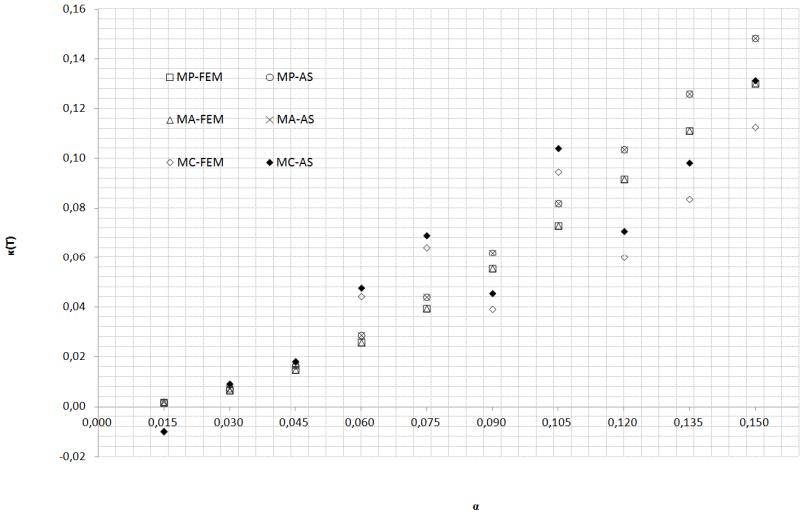

Fig. 8. Kurtosis of the temperature T_A as a function of input coefficient of variation α

Rys. 8. Kurtoza temperatury TA w funkcji wejściowego współczynnika wariancji α

Expected values calculated via all the SFEM approaches show some underestimation of the analytical results. Additionally, these expectations decrease all moderately together with an increase of the input coefficient of variation. Systematically simulation, perturbation and integral methods return almost the same results with accidental numerical discrepancies. Coefficients of variation (Fig. 6) are all the same independently of the calculation method and all linearly increase together with $\alpha(k)$. This is a consequence of similarity of the temperatures histogram to the Gaussian bell-shaped curve detected before in Figs. 3-4. A comparison of skewness calculated via different methods (Fig. 7) is not so perfect but the principal trend remains the same – this is a nonlinear convex increase together with the input CoV through dominantly nonnegative values. Remarkably, the same range of all numerical results enable to conclude that the resulting PDF increases its positive skewness together with an increase of input random fluctuations. Similar observations concern the kurtosis of steady-state temperature (Fig. 8) – both FEM and series analytical solution return used in conjunction with any of the probabilistic method almost the same numbers here. One should remark that the total number of random trials applied for third and fourth order statistics (similarly to the analysis carried out by Wan and Karniadakis (2006) [9]) could be increased according to a remarkable asymptotic convergence of these estimators about their principal trends. Nevertheless, taking into account relatively small values of both skewness and kurtosis as well as constant ratio of the output versus input randomness level one needs to conclude that the final temperature has the probability density function very close to the Gaussian one.

5. Concluding remarks

The most important conclusion that can be drawn from this analysis is that the steady state temperature probabilistic characteristics are practically the same according to all the probabilistic methods included into this comparative study; this observation is almost independent of an input uncertainty in Gaussian heat conductivity coefficient. So that one may notice that the generalized stochastic perturbation technique implemented with the Least Squares Method with polynomial basis of an order subjected to statistical optimization overcomes most of numerical discrepancies and limitations of the Second Order Second Moment (SOSM) numerical technique, see Kamiński and Hien (1999) [10]. It enables for further time and computer effort savings, at least while studying Gaussian input uncertainties in transient heat transfer linear problems.

A precision of the generalized stochastic perturbation technique used in conjunction with the Finite Element Method for other probability distributions deserves some separate computational studies and comparative validation. It is very important to notice that the SFEM methodology proposed here enables for a very precise determination of higher order statistics of the temperatures during transient process with overall computational cost similar rather to the original

deterministic problem than the Monte-Carlo simulation scheme, even in its importance sampling version. Further implementation of this dual probabilistic technique towards SFEM modeling of transient heat transfer problems with state-dependent physical parameters should be provided. Its applicability to the solution of some inverse problems in thermodynamics is also notice worthy similarly to the research findings of Blackwell and Beck (2010) [11].

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DUALNA PROBABILISTYCZNA ANALIZA NIESTACJONARNEGO PRZEPŁYWU CIEPŁA PRZY UŻYCIU STOCHASTYCZNEJ METODY ELEMENTÓW SKOŃCZONYCH

S t r e s z c z e n i e

Głównym celem niniejszej pracy jest porównanie trzech różnych probabilistycznych metod numerycznych, tj. metody analitycznej, symulacyjnej, a także metody perturbacji, podczas rozwią-

zywania pewnego zagadnienia osiowo-symetrycznego, w którym współczynnik przewodnictwa ciepła jest Gaussowskim parametrem losowym. Porównanie takie jest przeprowadzone przy użyciu systemu Metody Elementów Skończonych *ABAQUS* (dla części deterministycznej rozwiązania), a także pakietu algebry komputerowej *MAPLE*, w którym zaimplementowano wszystkie procedury losowe. W pracy wyznacza się centralne momenty probabilistyczne do rzędu czwartego włącznie, tj. wartości oczekiwane, współczynniki wariancji, skośność i kurtozę, jak również odpowiednie histogramy – wszystkie one są wyznaczone w funkcji wejściowego współczynnika wariancji. Metoda perturbacji stochastycznej jest zaimplementowana przy użyciu rozwinięcia w szereg Taylora rzędu dziesiątego, a także z wykorzystaniem tradycyjnej Metody Najmniejszych Kwadratów. Metoda ta umożliwia wyznaczenie wielomianowej funkcji odpowiedzi, której rząd jest przedmiotem oddzielnej optymalizacji statystycznej. Otrzymane wyniki probabilistyczne pokazują bardzo dobrą zgodność wszystkich wyznaczanych charakterystyk losowych, co oznacza w praktyce, iż tradycyjna metoda symulacji może zostać zastąpiona przez metodę perturbacji stochastycznej w celu wielokrotnego zmniejszenia czasu oraz mocy obliczeniowej.

Słowa kluczowe: przepływ ciepła; Stochastyczna Metoda Elementów Skończonych; metoda symulacji Monte-Carlo; metoda perturbacji stochastycznej

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